Robust Stabilization for Time-varying Delay Uncertain Chaotic Systems with Delay-Dependence

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ABSTRACT

In this paper, the problem of stabilization criteria for a class of linear time-delay uncertain chaotic systems is derived. Based on the Lyapunov-Krasovskii functional combining with linear matrix inequality (LMI) techniques and Leibniz-Newton formula, delay-dependent stabilization criteria are derived for the existence of a state feedback controller, which ensures asymptotic stability of the closed systems for all admissible uncertainties. Furthermore, we try to transform the LMI feasible problem into the equivalent generalized eigenvalues problem (GEVP). Such that the global solution, namely the maximum upper bound on the admissible delay, can be determine by using the LMI toolbox in Matlab.

KEYWORDS: Chaotic Systems, integral inequality approach (IIA), robust stability, linear matrix inequality ( LMI)

1. Introduction

Time-delay occurs in many physical systems such as population dynamics, neural networks, automatic control systems, biology, economy, and so on. The time-delay is frequently a source of instability and performance deterioration. Therefore, stability analysis and controller synthesis for time-delay system have been one of the most challenging issues. Furthermore, since time-delay systems have infinite-dimensionality, it is difficult to apply conventional well-known stability theory. It is well-known that there exist time delays in the information processing of neurons due to various reasons. For example, time delays can be caused by the finite switching speed of amplifier circuits in neural networks or deliberately introduced to achieve
tasks of dealing with motion-related problems, such as moving image processing. Time delays in the neural networks make the dynamic behaviors become more complicated, and may destabilize the stable equilibriums and admit periodic oscillation, bifurcation and chaos. Therefore, considerable attention has been paid on the study of delay systems in control theory and a large body of work has been reported in the literature [1-12] and the references therein.

In 1963, Lorenz found the first chaotic attractor in a simple three-dimensional autonomous system. So far there are many researchers who studied the chaos theory. During the last decades, dynamic chaos theory has been deeply studied and applied to many fields extensively, such as secure communications, optical system, biology and so forth. Since Mackey and Glass [5] first found chaos in time delay systems, there has been increasing interest in time delay chaotic systems. One of the most frequent objectives consists in the stabilization of chaotic behaviors to one of unstable fixed points or unstable periodic orbits embedded within a chaotic attractor. That is, to design a suitable controller that guarantees the closed-loop system dynamics converges to the fixed point or periodic orbit. Guan et al. [2] and Sun [11] have investigated the controller design problem of a class of time-delay chaotic systems using the famous OGY-method [6]. Recently, Park and Kwon [9] used a parameterized neutral model transformation approach and an integral inequality for stabilization of a class of time-delay chaotic systems. They proposed a standard feedback controller and a delayed feedback control which are conservative than the results of [1] and [11].

Chaos, a very interesting nonlinear phenomenon, has been intensively studied in the last three decades. Chaotic behavior has been observed in numerous natural systems such as in physics (laser technology, plasma), chemistry, biology, ecology, mechanical engineering and chemistry, and etc. Since chaotic behavior usually results in irregular oscillations. Therefore, chaos control has received much attention during the past decades. To overcome the conservatism of the previous results, we are going to propose a novel Lyapunov–Krasovskii functional and do not use any model transformation technique. The delay-dependent controller design method obtained stabilizes the system to an unstable fixed point. Integral inequality approach (IIA) is used to reduce the conservativeness of the proposed controller. The numerical example shows that the proposed method is of less conservativeness than the previous results.

In this paper, new stabilization criteria for a class of linear time-delay uncertain
chaotic systems based on Lyapunov functional will be derived. By developing a integral inequality approach, the information of the delayed plant states can be taken into full consideration, and new delay-dependent sufficient stability criteria are obtained in terms of linear matrix inequalities (LMIs) which can be easily solved by various optimization algorithms. Since the delay term is concerned more exactly, less conservative results are presented. Furthermore, the paper gives the stabilization and optimality algorithm to solve the optimal result. Finally, a numerical example is given to demonstrate the feasibility and solve the generalized eigenvalue minimization problem (GEVP) of our proposed approach.

2 Problem formulation and preliminaries

Consider a time-delay chaotic system represented by the following equation:

\[ \dot{x}(t) = Ax(t) + Bx(t - h_1) + u(t) + f_1(x(t), t) + f_2(x(t - h_1), t) \]  

(1)

where \( x(t) \in R^n \) is the state vector, \( u(t) \in R^n \) is the control input vector, \( A, B \in R^{n\times n} \) are constant system matrices representing the linear parts of the system, \( f_1, f_2 \in R^{n\times n} \) are nonlinear parts of the system, and \( h_1 > 0 \) is the constant time-delay. Suppose that the chaotic system (1) has an unstable fixed point or an unstable periodic orbit \( x(t) \), and is currently in a chaotic state. Then the purpose of this paper is to control the system asymptotically converges towards \( x(t) \) with only extremely small force \( u(t) \).

The first control method is the SFC approach whose control input is chosen as

\[ u(t) = K(x(t) - x(t)) \]  

(2)

The second control method in this paper is the DFC approach whose control input is presented as

\[ u(t) = K[x(t - h_2) - x(t)] \]  

(3)

where \( K \) is an adjustable coefficient of the controller, and \( x(t) \) is the desired fixed point. \( h_2 > 0 \) is the constant feedback time-delay.

Note that the unstable fixed point \( \bar{x}(t) = x(t) = \text{constant} \) satisfies the following equation:
\[
\dot{x}(t) = A\dot{x}(t) + Bx(t-h_1) + u(t) + f_1(x(t), t) + f_2(x(t-h_1), t)
\]  
(4)

Define \( e(t) = x(t) - \tilde{x}(t) \). Then, subtracting (4) from (1) with the control law (2) yields the following error dynamics:

\[
\dot{e}(t) = Ae(t) + Be(t-h_1) + F_1(e(t), t) + F_2(e(t-h_1), t) - Ke(t) \\
= (A-K)e(t) + Be(t-h_1) + F_1(e(t), t) + F_2(e(t-h_1), t)
\]  
(5)

And subtracting (4) from (1) with the control law (3) yields the following error dynamics:

\[
\dot{e}(t) = Ae(t) + Be(t-h_1) + F_1(e(t), t) + F_2(e(t-h_1), t) + K[e(t-h_2) - e(t)] \\
= (A-K)e(t) + Be(t-h_1) + Ke(t-h_2) + F_1(e(t), t) + F_2(e(t-h_1), t)
\]  
(6)

where \( F_i(e(t), t) = f_i'(e(t) + \tilde{x}(t), t) - f_i'(x(t), t) \), and

\[
F_2(e(t-h_1), t) = f_2'(e(t-h_1) + \tilde{x}(t-h_1), t) - f_2'(\tilde{x}(t-h_1), t).
\]

For the error systems (5) or (6), since zero is a fixed pointed of \( F_1(e(t), t) + F_2(e(t-h_1), t) \) we have a Taylor expansion

\[
F_i(e(t), t) + F_2(e(t-h_1), t) = \beta_0 + [H.O.T.]_1 + \beta_1 + [H.O.T.]_2
\]  
(7)

\( \beta_0 = F_1(e(t), t), \beta_1 = F_2(e(t-h_1), t), \) [H.O.T.]_1 is higher order term in e(t), [H.O.T.]_2 is higher order term in e(t - h_1) and \( F_i \) stands for the time derivative of \( F_i (i = 1, 2) \).

Then, the control goal is to force \( \|e\| \to 0 \) as \( t \to \infty \).

For the error system, from the OGY-method, our control objective is the zero fixed point, so we can only consider the linearized part near zero point. Rewrite the local error system (6) as follows:

\[
\dot{e}(t) = (A-K + \beta_0 J) e(t) + (A_d + \beta_1 J) e(t-h_1) + Ke(t-h_2)
\]  
(8)

Our goals are to establish a sufficient condition on delay-dependent stability and to give estimates of \( h_d, h_1 \) and \( h_2 \). The following technical Lemma 1 of integral
inequality approach will be used in the sequel.

**Lemma 1 [4]:** For any positive semi-definite matrices

\[
X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23} & X_{33} \end{bmatrix} \geq 0
\]  
(9a)

the following integral inequality holds

\[
-\int_{t-h}^{t} \dot{x}^T(s)X_{33}\dot{x}(s)ds
\leq \int_{t-h}^{t} \begin{bmatrix} x^T(t) & x^T(t-h) & \dot{x}^T(s) \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(s) \end{bmatrix} ds
\]  
(9b)

**Lemma 2 [1].** The following matrix inequality

\[
\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0
\]  
(10a)

where \(Q(x) = Q^T(x), R(x) = R^T(x)\) and \(S(x)\) depend affine on \(x\), is equivalent to

\[
R(x) < 0 \quad (10b)
\]

\[
Q(x) < 0 \quad (10c)
\]

and

\[
Q(x) - S(x)R^{-1}(x)S^T(x) < 0 \quad (10d)
\]

Theorem 1 presents delay-dependent result in terms of LMIs and expresses relationships between the terms of the Leibniz–Newton formula.

**Theorem 1:** Given positive scalars \(h_1, h_2\), and \(h_d\), the time-delay chaotic systems (1) is stochastically stable if there exist symmetry positive-definite matrices \(W = W^T > 0\), \(U_i = U_i^T > 0\), \(S_i = S_i^T > 0\) \((i = 1, 2)\), positive semi-definite matrices

\[
M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{bmatrix} \geq 0, \quad N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}^T & N_{22} & N_{23} \\ N_{13}^T & N_{23} & N_{33} \end{bmatrix} \geq 0
\]

and a matrix \(L\) with appropriate dimensions such that the following holds:
\[ \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} \\ \Psi_{12} & \Psi_{22} & \Psi_{23} & \Psi_{24} & \Psi_{25} \\ \Psi_{13} & \Psi_{23} & \Psi_{33} & \Psi_{34} & \Psi_{35} \\ \Psi_{14} & \Psi_{24} & \Psi_{34} & \Psi_{44} & \Psi_{45} \\ \Psi_{15} & \Psi_{25} & \Psi_{35} & \Psi_{45} & \Psi_{55} \end{bmatrix} < 0 \] (11a)

and

\[ \begin{align*}
W - \bar{X}_{33} & \geq 0 \tag{11b} \\
W - \bar{Y}_{33} & \geq 0 \tag{11c}
\end{align*} \]

\[ \Psi_{11} = W(A + \beta_0 I)^T + (A + \beta_0 I)W - L - L^T + U_1 + U_2 + h_1 M_{11} + M_{13}^T + M_{13} + h_2 N_{11} + N_{13} + N_{13}^T, \]

\[ \Psi_{12} = (B + \beta I)W + h_1 M_{12} - M_{13} + M_{23}, \Psi_{13} = Y + h_2 N_{12} - N_{13} + N_{13}^T, \]

\[ \Psi_{14} = W(A + \beta_0 I)^T - Y^T, \Psi_{15} = W(A + \beta_0 I)^T - Y^T, \Psi_{22} = -U_1 + h_1 N_{22} - N_{23} - N_{23}^T, \]

\[ \Psi_{22} = W(B + \beta I)^T, \Psi_{25} = W(B + \beta I)^T, \Psi_{34} = -U_2 + h_2 N_{22} - N_{23} - N_{23}^T, \]

\[ \Psi_{34} = L^T, \Psi_{35} = L^T, \Psi_{44} = -h_1 S_1, \Psi_{55} = -h_2 S_2, \Psi_{23} = \Psi_{45} = 0. \]

Then, the time-delay chaotic system (1) is asymptotically stable for with allowable time delay \( h \) under the state feedback control law and \( K = LW^{-1} \) is a stabilizing gain.

**Proof:** Choose a Lyapunov-Krasovskii functional candidate as

\[ V(t) = V_1(t) + V_2(t) + V_3(t) \tag{12} \]

where

\[ V_1(t) = e^T(t)Pe(t) \]

\[ V_2(t) = \int_{t-h}^{t} e^T(s)Q_1 e(s)ds + \int_{t-h_2}^{t} e^T(s)Q_2 e(s)ds \]

\[ V_3(t) = \int_{t-h}^{t} \int_{t-\theta}^{\theta} e^T(s)R_1 e(s)d\theta + \int_{t-h_2}^{t} \int_{t-\theta}^{\theta} e^T(s)R_2 e(s)d\theta \]

Taking the derivative of \( V(x) \) for \( t > 0 \) along the trajectory of (8) yields that

\[ \dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t). \tag{13} \]

First the derivative of \( V_1(t) \) is
\[ \hat{V}_1(t) = \hat{e}^T(t)P\hat{e}(t) + \hat{e}^T(t)P\hat{e}(t) \]
\[ = [(A - K + \beta_d)\hat{e}(t) + (B + \beta_d)\hat{e}(t - h_1) + Ke(t - h_2)]^T Pe(t) + e^T(t)P[(A - K + \beta_d)\hat{e}(t) + (B + \beta_d)\hat{e}(t - h_1) + Ke(t - h_2)] \]
\[ = \hat{e}^T(t)[(A - K + \beta_d)\hat{e}(t) + P(A - K + \beta_d)] + e^T(t)Ke(t - h_2) \]
\[ + e^T(t)P(B + \beta_d)\hat{e}(t - h_1) + e^T(t)PK\hat{e}(t - h_2) \]
\[ + e^T(t - h_2)(B + \beta_d)\hat{e}(t - h_1)P(t) + \hat{e}^T(t - h_2)K^T Pe(t) + \hat{e}^T(t - h_2)K^T Pe(t) \] (14)

Second, we get the bound of \( V_2(t) \) as

\[ \hat{V}_2(t) = \hat{e}^T(t)(Q_1 + Q_2)\hat{e}(t) - \hat{e}^T(t - h_1)Q_2\hat{e}(t - h_1) - \hat{e}^T(t - h_2)Q_2\hat{e}(t - h_2) \] (15)

Finally, the bound of \( V_3(t) \) is as follows:

\[ \hat{V}_3(t) = \hat{e}^T(t)(h_1R_1 + h_2R_2)\hat{e}(t) - \int_{t-h_1}^{t} \hat{e}^T(s)R\hat{e}(s)ds - \int_{t-h_1}^{t} \hat{e}^T(s)R_2\hat{e}(s)ds \] (16)

From (14)-1(16), we have

\[ \hat{V}(t) = \hat{V}_1(t) + \hat{V}_2(t) + \hat{V}_3(t) \]
\[ = \hat{e}^T(t)[(A - K + \beta_d)\hat{e}(t) + P(A - K + \beta_d)] + e^T(t)Ke(t - h_2) \]
\[ + e^T(t)P(B + \beta_d)\hat{e}(t - h_1) + e^T(t)PK\hat{e}(t - h_2) \]
\[ - \hat{e}^T(t - h_1)Q_2\hat{e}(t - h_1) + e^T(t - h_2)K^T Pe(t) - \hat{e}^T(t - h_1)Q_2\hat{e}(t - h_1) \]
\[ + e^T(t)P(B + \beta_d)\hat{e}(t - h_1) + e^T(t - h_2)K^T Pe(t) - \hat{e}^T(t - h_2)Q_2\hat{e}(t - h_2) \]
\[ - \int_{t-h_1}^{t} \hat{e}^T(s)R\hat{e}(s)ds - \int_{t-h_1}^{t} \hat{e}^T(s)R_2\hat{e}(s)ds \]
\[ - \int_{t-h_1}^{t} \hat{e}^T(s)(R_2 - Y_3s)\hat{e}(s)ds - \int_{t-h_1}^{t} \hat{e}^T(s)X_3\hat{e}(s)ds - \int_{t-h_1}^{t} \hat{e}^T(s)Y_3\hat{e}(s)ds \] (17)

Using the Leibniz-Newton formula \( x(t) - x(t - h) = \int_{t-h}^{t} \dot{x}(s)ds \), and Lemma 2, we obtain

\[ -\int_{t-h_1}^{t} \hat{e}^T(s)X_3\hat{e}(s)ds \leq \int_{t-h_1}^{t} \begin{bmatrix} \hat{e}^T(t) & \hat{e}^T(t - h_1) & \hat{e}^T(s) \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13} & X_{23} & 0 \end{bmatrix} \begin{bmatrix} \hat{e}(t) \\ \hat{e}(t - h_1) \\ \hat{e}^T(s) \end{bmatrix} ds \]
\begin{align*}
&\leq e^T(t)h_1X_1\xi(t) + e^T(t)h_1X_1\xi(t-h_1) + e^T(t)X_{12}\int_{t-h_1}^t \hat{\epsilon}(s)ds + e^T(t-h_1)h_2X_{12}^T\xi(t) \\
&+ e^T(t-h_1)h_1X_2\xi(t-h_1) + e^T(t-h_1)X_{23}\int_{t-h_1}^t \hat{\epsilon}(s)ds \\
&+ \int_{t-h_1}^t \hat{\epsilon}(s)ds X_2^T\xi(t) + \int_{t-h_1}^t \hat{\epsilon}(s)ds X_2^T\xi(t-h_1) \\
&= e^T(t)[h_1X_{11} + X_{12}^T + X_{13}]\xi(t) + e^T(t)[h_1X_{12} - X_{13} + X_{23}^T]\xi(t-h_1) \\
&+ e^T(t-h_1)[h_1X_{12}^T - X_{13}^T + X_{23}]\xi(t-h_1)
\end{align*}

Similarly, we have

\begin{align*}
&-\int_{t-h_2}^t \hat{\epsilon}^T(s)Y_3\hat{\epsilon}(s)ds \\
&\leq e^T(t)(h_2Y_{11} + Y_{12} + Y_{13}^T)\xi(t) - e^T(t)(h_2Y_{12} - Y_{13} + Y_{23}^T)e(t-h_2) \\
&+ e^T(t-h_2)(h_2Y_{12} - Y_{13} + Y_{23}^T)e(t-h_2)(h_2Y_{22} - Y_{23} + Y_{23}^T)e(t-h_2)
\end{align*}

Evaluating \( \hat{\epsilon}^T(t)(h_1R_1 + h_2R_2)\hat{\epsilon}(t) \) along solution to (8), gives as follows:

\begin{align*}
\hat{\epsilon}^T(t)(h_1R_1 + h_2R_2)\hat{\epsilon}(t) \\
= &\left[(A - K + \beta_dJ)e(t) + (B + \beta_fJ)e(t-h_1) + Ke(t-h_2)\right]^{T}(h_1R_1 + h_2R_2) \\
&\left[(A - K + \beta_dJ)e(t) + (B + \beta_fJ)e(t-h_1) + Ke(t-h_2)\right] \\
= &e^T(t)(A - K + \beta_dJ)^T(h_1R_1 + h_2R_2)(A - K + \beta_dJ)e(t) \\
&+ e^T(t)(A - K + \beta_dJ)^T(h_1R_1 + h_2R_2)(B + \beta_fJ)e(t-h_1) \\
&+ e^T(t)(A - K + \beta_dJ)^T(h_1R_1 + h_2R_2)Ke(t-h_2) \\
&+ e^T(t-h_1)(B + \beta_fJ)^T(h_1R_1 + h_2R_2)(A - K + \beta_dJ)e(t) \\
&+ e^T(t-h_1)(B + \beta_fJ)^T(h_1R_1 + h_2R_2)(B + \beta_fJ)e(t-h_1) \\
&+ e^T(t-h_1)(B + \beta_fJ)^T(h_1R_1 + h_2R_2)Ke(t-h_2) \\
&+ e^T(t-h_2)K^T(h_1R_1 + h_2R_2)(A - K + \beta_dJ)e(t) \\
&+ e^T(t-h_2)K^T(h_1R_1 + h_2R_2)(B + \beta_fJ)e(t-h_1) \\
&+ e^T(t-h_2)K^T(h_1R_1 + h_2R_2)Ke(t-h_2)
\end{align*}

Substituting the above equations (18)-(20) into (13), we obtain

\begin{align*}
\dot{V}(t) &\leq \xi^T(t)\Omega\xi(t) - \int_{t-h_1}^t \hat{\epsilon}^T(s)(R_1 - X_{33})\hat{\epsilon}(s)ds - \int_{t-h_2}^t \hat{\epsilon}^T(s)(R_2 - Y_{33})\hat{\epsilon}(s)ds
\end{align*}

where \( \xi^T(t) = \left[\hat{\epsilon}^T(t) \quad \hat{\epsilon}^T(t-h_1) \quad \hat{\epsilon}^T(t-h_2)\right] \) and
\[
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} \\
\Omega_{12}^T & \Omega_{22} & \Omega_{23} \\
\Omega_{13} & \Omega_{23} & \Omega_{33}
\end{bmatrix}
\]

\[
\Omega_{11} = (A - K + \beta J)^T P + P(A - K + \beta \theta J) + Q_1 + Q_2 + h_1X_{11} + X_{13} + X_{13}^T \\
+ h_2Y_{12} + Y_{13} + Y_{13}^T + (A - K + \beta \theta J)^T (h_1R_1 + h_2R_2)(A - K + \beta \theta J)
\]

\[
\Omega_{12} = P(B + \beta J) + h_1X_{12} - X_{13} + X_{23} + (A - K + \beta \theta J)^T (h_1R_1 + h_2R_2)(B + \beta J) \\
\Omega_{13} = PK + h_2Y_{12} - Y_{13} + Y_{13}^T + (A - K + \beta \theta J)^T (h_1R_1 + h_2R_2)K \\
\Omega_{22} = -Q_1 + h_1X_{22} - X_{23} - X_{23}^T + (B + \beta J)^T (h_1R_1 + h_2R_2)(B + \beta J) \\
\Omega_{23} = (B + \beta J)^T (h_1R_1 + h_2R_2)K \\
\Omega_{33} = -Q_2 + h_2Y_{22} - Y_{23} - Y_{23}^T + K^T (h_1R_1 + h_2R_2)K
\]

From Equation (1) and the Schur complement, it is easy to see that \( \dot{V}(t) < 0 \) holds if \( R_1 - X_{33} \geq 0, R_2 - Y_{33} \geq 0 \). If LMIs (7) are feasible, system (1) is asymptotically stable.

\[
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} \\
\Xi_{12}^T & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} \\
\Xi_{13} & \Xi_{23} & \Xi_{33} & \Xi_{34} & \Xi_{35} \\
\Xi_{14} & \Xi_{24} & \Xi_{34} & \Xi_{44} & \Xi_{45} \\
\Xi_{15} & \Xi_{25} & \Xi_{35} & \Xi_{45} & \Xi_{55}
\end{bmatrix} < 0
\]  
(22)

where

\[
\Xi_{11} = (A - K + \beta J)^T P + P(A - K + \beta \theta J) + Q_1 + Q_2 + h_1X_{11} + X_{13} + h_2Y_{11} + Y_{13} + Y_{13}^T, \\
\Xi_{12} = P(B + \beta J) + h_1X_{12} - X_{13} + X_{23}, \Xi_{13} = PK + h_2Y_{12} - Y_{13} + Y_{13}^T, \\
\Xi_{14} = h_1(A - K + \beta \theta J)^T R_1, \Xi_{15} = h_2(A - K + \beta \theta J)^T R_2, \Xi_{22} = -Q_1 + h_1X_{22} - X_{23} - X_{23}^T, \\
\Xi_{23} = h_1(B + \beta J)^T R_1, \Xi_{25} = h_2(B + \beta J)^T R_2, \Xi_{33} = -Q_2 + h_2Y_{22} - Y_{23} - Y_{23}^T, \Xi_{34} = h_1K^T R_1, \\
\Xi_{35} = h_2K^T R_2, \Xi_{44} = -h_1R_1, \Xi_{55} = -h_2R_2, \Xi_{23} = \Xi_{45} = 0.
\]

Pre- and post- multiplying both sides of (22) by \( \text{diag}\{P^{-1}, P^{-1}, P^{-1}, R_1^{-1}, R_2^{-1}\} \) and letting

\[
W = P^{-1}, U_1 = P^{-1}Q_1P^{-1}, U_2 = P^{-1}Q_2P^{-1}, M_\theta = P^{-1}X_\theta P^{-1}, N_\theta = P^{-1}Y_\theta P^{-1}, \\
S_1 = R_1^{-1}, S_2 = R_2^{-1}, K = W^{-1}Y, \quad K = LW^{-1}, \quad \begin{bmatrix} R_1^{-1} & P^{-1} \end{bmatrix} \begin{bmatrix} R_1 \cr -X_{33} \end{bmatrix} P^{-1} = W - Y_{33}, \quad \begin{bmatrix} R_2^{-1} \\
- Y_{33}
\end{bmatrix} P^{-1} = W - Y_{33},
\]

yields (11). This completes the proof. \( \square \)
3. Example

Consider a delayed chaotic system described by [11]

\[
\dot{e}(t) = (A - K + \beta_0 I_e) e(t) + (A_d + \beta_0 I) e(t - h_1) + Ke(t - h_2)
\]

where \( A = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \beta_0 = 0, \beta_0 = -1. \]

Solution. First of all, let \( h_1 = 1, h_2 = 2 \) by using LMI Toolbox in MATLAB (with accuracy 0.01), then solutions of LMI given in (11) are found to be

\[
W = \begin{bmatrix}
46.8446 & -12.8312 \\
-12.8312 & 47.1863
\end{bmatrix}, U_1 = \begin{bmatrix}
8.0953 & -5.1604 \\
-5.1604 & 13.1923
\end{bmatrix}, U_2 = \begin{bmatrix}
32.2845 & 2.2427 \\
2.2427 & 56.5250
\end{bmatrix},
\]

\[
S_1 = \begin{bmatrix}
9.7729 & 0.9148 \\
0.9148 & 3.0593
\end{bmatrix}, S_2 = \begin{bmatrix}
5.9732 & 0.3512 \\
0.3512 & 2.6327
\end{bmatrix}, M_{11} = \begin{bmatrix}
6.4164 & -2.2769 \\
-2.2769 & 8.1578
\end{bmatrix}, M_{12} = \begin{bmatrix}
-2.5388 & -0.9234 \\
-1.3312 & -0.4378
\end{bmatrix}, M_{13} = \begin{bmatrix}
-2.5133 & -0.4241 \\
-1.2296 & -0.2743
\end{bmatrix}, M_{22} = \begin{bmatrix}
6.1674 & -1.0375 \\
-1.0375 & 5.8890
\end{bmatrix}, M_{23} = \begin{bmatrix}
2.5052 & 0.3538 \\
0.8213 & 0.3190
\end{bmatrix}, M_{33} = \begin{bmatrix}
5.7515 & 0.6282 \\
0.6282 & 1.5650
\end{bmatrix},
\]

\[
N_{12} = \begin{bmatrix}
-1.1088 & -1.0095 \\
-0.6016 & 0.8917
\end{bmatrix}, N_{13} = \begin{bmatrix}
-0.8801 & -0.1219 \\
0.0493 & -0.0084
\end{bmatrix}, N_{22} = \begin{bmatrix}
2.8109 & -0.0848 \\
-0.0848 & 2.1490
\end{bmatrix}, N_{23} = \begin{bmatrix}
0.8544 & 0.0849 \\
0.0821 & 0.0760
\end{bmatrix}, N_{33} = \begin{bmatrix}
3.1814 & 0.1977 \\
0.1977 & 1.3216
\end{bmatrix}.
\]

The results validates the fact that the formulation of stability conditions in LMI frame work by delay-dependent approach for ascertaining stability of time-delay systems gives conservative results compared to delay-independent treatment of the delays. This paper has the potential to enable us to obtain less conservative results than those obtained by [2, 5, 8, 10].

4. Conclusion

In this paper delay-dependent stability criteria have been developed for linear time-delay uncertain chaotic systems. In this method, integral inequality matrices express the relationships between the terms in the Leibniz-III Newton formula so that a solution can be obtained by solving linear matrix inequalities (LMIs) which make the method less conservative than those employing fixed weighting matrices. Finally,
a numerical example is given to demonstrate the feasibility and solve the generalized eigenvalue minimization problem (GEVP) of our proposed approach.

References