Robust exponential stability analysis for delayed neural networks with time-varying delay

Liu Pin-Lin

*Corresponding author, Department of Automation Engineering Institute of Mechatronoptic System, Chienkuo Technology University, Changhua, 500, Taiwan, R.O.C., Tel: 886-7111155, Fax: 886-7111129, Email:lpl@cc.ctu.edu.tw

ABSTRACT

In this paper, we investigate the problem of robust stability for delayed neural network systems with time-varying delay. By means of Lyapunov-Krasovskii functional approach, integral inequality approach and linear matrix inequality (LMI) concept, some new stability conditions are derived for above systems. There are three main parts concerning our research results. The first result is to propose both delay-independent and delay-dependent criteria for guaranteeing the asymptotic stability of delayed neural network systems with constant parameters and discrete delay. The second result is to present several new delay-dependent criteria for testing the exponential stability of delayed neural network systems with time-varying delays. The third result is to provide sufficient conditions for ensuring both asymptotic stability and exponential stability of delayed neural network systems with time-varying delays. In the results, we do not assume that the network's activation functions are with the property of sigmoid function. The purpose of introducing the integral inequality approach is to improve the results on bounds of the inner product of two vectors. Our results do not need the solution of Lyapunon equation or Riccati equation. Compared with existing results in the literature, our method is shown to be superior to other ones. Numerical examples are given to demonstrate the effectiveness of the proposed approach. Besides, our approach can be also applied to the stability problem for large-scale neural network systems with variable parameters and delays.

KEYWORDS

Neural network; distributed delay; integral inequality approach; linear matrix inequality
1. Introduction

It is well known that neural networks have been extensively studied over the past few decades and have been successfully applied to many areas such as signal processing, static processing, pattern recognition, combinatorial optimization, and so on [2]. So, it is important to study the stability of neural networks. In biological and artificial neural networks, time delays often arise in the processing of information storage and transmission. Recently, a considerable number of sufficient conditions on the existence, uniqueness and global asymptotic stability of equilibrium point for neural networks with constant delays or time-varying delays were reported under some assumptions, for example, see [4, 5, 7, 9, 10, 12, 13, 14, 15] and references therein. In the design of delayed neural networks (DNNs), however, one is interested not only in the global asymptotic stability of the neural network, but also in some other performances. In particularly, it is often desirable that the neural network converges fast enough in order to achieve fast response [13]. It is well known that exponential stability gives a fast convergence rate to the equilibrium point. Therefore, some researchers studied the exponential stability analysis problem for time delay systems with constant delays or time-varying delays, and a great number of results on this topic have been given in the literature, for example, see [4, 8, 10, 11, 12, 13, 14, 15] and references therein. The literature [14] first uses the linear matrix inequality (LMI) technique to guarantee the global exponential stability of the DNNs with time-varying delays, but it has the restriction that the change rate of time-varying delay satisfies \( \dot{h}(t) < 1 \). In practical implementation of neural networks, uncertainties are inevitable in neural networks because of the existence of modelling errors and external disturbance. It is important to ensure the neural networks system is stable under these uncertainties. Both time delays and uncertainties can destroy the stability of neural networks in electronic implementation. Therefore, it is of great theoretical and practical importance to investigate the robust stability for delayed neural networks with uncertainties [12].

Recently, a free-weighting matrix approach [4] has been employed to study the exponential stability problem for neural networks with a time-varying delay [5]. However, as mentioned in [6], some useful terms in the derivative of the Lyapunov functional were ignored in [5, 7, 15]. For example, the derivative of 
\[
\int_{-h}^{0} \int_{t-h}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta \\
\text{with } 0 \leq h(t) \leq \overline{h}
\]
was estimated as 
\[
\dot{x}^T(t)hZ\dot{x}(t)
\]
\[-\int_{-h(t)}^{t} \dot{x}(s)Z\dot{x}(s)ds\] and the negative term \[-\int_{-h(t)}^{t} \dot{x}(s)Z\dot{x}(s)ds\] was ignored in [15], which may lead to considerable conservativeness. Although, in [5] and [7], the negative term \[-\int_{-h(t)}^{t} \dot{x}(s)Z\dot{x}(s)ds\] was retained, the other term \[-\int_{-h(t)}^{t} \dot{x}(s)Z\dot{x}(s)ds\] was ignored, which may also lead to considerable conservativeness. On the other hand, if the free-weighting method introduces too many free-weighting matrices in the theoretical derivation, some of them sometimes have no effect on reducing the conservatism of the obtained results; on the contrary, they mathematically complicate the system analysis and consequently lead to a significant increase in the computational demand [9]. How to overcome the aforementioned disadvantages of the integral inequality approach is an important research topic in the delay-dependent related problem and also motivates the work of this paper on exponential stability analysis. Furthermore, the restriction of \( \dot{h}(t) < 1 \) is released in the proposed scheme.

Motivated by the above discussions, the objective of this paper is to study the global exponential robust stability of the delayed neural network with time-varying delays. The proposed stability criteria in this paper are in terms of linear matrix inequalities (LMI), which are easy to check by recently developed algorithms solving LMIs. Furthermore, examples with simulation are given to show that the proposed stability criteria are less conservative than some recently known ones in the literature.

**Notation:** Throughout this correspondence paper, \( N^T \) stands for the transpose of the matrix \( N \), \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space, \( P > 0 \) means that the matrix \( P \) is positive definite, \( I \) is an appropriately dimensioned identity matrix, \( (\text{diag} \ldots) \) denotes a block diagonal matrix.

### 2. Problem Formulations and Preliminaries

Consider continuous neural networks with time-varying delays can be described by the following state equations:

\[
\dot{u}(t) = -(c_i + \Delta c_i)u_i(t) + \sum_{j=1}^{n} (a_{ij} + \Delta a_{ij})f_j(u_j(t)) + \sum_{j=1}^{n} (b_{ij} + \Delta b_{ij})f_j(u_j(t-h(t))) + J, \tag{1}
\]

or equivalently

\[
\dot{u}(t) = -(C + \Delta C(t))u(t) + (A + \Delta A(t))f(u(t)) + (B + \Delta B(t))f(u(t-h(t))) + J, \tag{2}
\]
where $u(t) = [u_1(t), u_2(t), ..., u_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, $f(u(t)) = [f_1(u_1(t)), f_2(u_2(t)), ..., f_n(u_n(t))]^T \in \mathbb{R}^n$ is the activation functions, $f(u(t-h(t))) = [f_1(u_1(t-h(t))), f_2(u_2(t-h(t))), ..., f_n(u_n(t-h(t)))]^T \in \mathbb{R}^n$, $J = [J_1, J_2, ..., J_n]^T$ is a constant input vector, $C = \text{diag}(c_i)$ is a positive diagonal matrix, $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ are the interconnection matrices representing the weight coefficients of the neurons. The matrices $\Delta C(t), \Delta A(t)$ and $\Delta B(t)$ are the uncertainties of the system and have the form

$$
[\Delta C(t) \ \Delta A(t) \ \Delta B(t)] = DF(t)[E_c \ E_a \ E_b],
$$

where $D, E_c, E_a$, and $E_b$ are known constant real matrices with appropriate dimensions and $F(t)$ is an unknown matrix function with Lebesgue-measurable elements bounded by

$$
F^T(t)F(t) \leq I,
$$

∀t,

(5)

where $I$ is an appropriately dimensioned identity matrix.

The time delay $h(t)$ is a time varying differentiable function that satisfies

$$
0 \leq h(t) \leq h, \quad \dot{h}(t) \leq h_d, \quad \forall t \geq 0,
$$

(6)

where $h$ and $h_d$ are constants.

Throughout this paper, it is assumed that each of the activation functions $f_j(j = 1, 2, ..., n)$ possess the following condition

$$
0 \leq \frac{f_j(u) - f_j(v)}{u - v} \leq k_j, \quad u \neq v \in \mathbb{R}, i = 1, 2, ..., n,
$$

(7)

where $k_j(i = 1, 2, ..., n)$ are positive constants.

Next, the equilibrium point $u^* = [u_1^*, ..., u_n^*]^T$ of system (1) is shifted to the origin through the transformation $x(t) = u(t) - u^*$, then system (1) can be equivalently
written as the following system

\[
\dot{x}(t) = -(C + \Delta C(t))x(t) + (A + \Delta A(t))g(x(t)) + (B + \Delta B(t))g(x(t - h(t))),
\]

where \(x(t) = [x_1(t), \cdots, x_n(t)]^T\), \(g(x(t)) = [g_1(x(t)), \cdots, g_n(x(t))]^T\),
\(g_i(x_i(t)) = f_i(x_i(t) + u_i) - f_i(u_i)\), \(i = 1, 2, \ldots, n\). It is obvious that the function \(g_j(\cdot)(j = 1, 2, \ldots, n)\) satisfies the following condition,

\[
0 \leq \frac{g_i(x_i)}{x_i} \leq k_i, \quad g_i(0) = 0, \quad \forall x_i \neq 0, i = 1, 2, \ldots, n,
\]

which is equivalent to

\[
g_i(x_i)(g_i(x_i) - k_i x_i) \leq 0, \quad g_i(0) = 0, \quad \forall x_i \neq 0, i = 1, 2, \ldots, n.
\]

Now we are stating the following lemmas which will be more useful in the sequel.

**Lemma 1 [8, 9].** For any semi-positive definite matrices

\[
X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13}^T & X_{23}^T & X_{33}
\end{bmatrix} \geq 0,
\]

the following integral inequality holds

\[
-\int_{t-h(t)}^{t} \dot{x}(s)X_{33}\dot{x}(s)ds \leq \int_{t-h(t)}^{t} \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13}^T & X_{23}^T & 0
\end{bmatrix} \begin{bmatrix}
x(t) \\
x(t-h(t)) \\
\dot{x}(s)
\end{bmatrix} ds.
\]

Secondary, we introduce the following Schur complement which is essential in the
proofs of our results.

**Lemma 2 [1].** The following matrix inequality

\[
\begin{bmatrix}
Q(x) & S(x) \\
S^T(x) & R(x)
\end{bmatrix} < 0,
\]

where \( Q(x) = Q^T(x), R(x) = R^T(x) \) and \( S(x) \) depend on affine on \( x \), is equivalent to

\[
R(x) < 0,
\]

(12b)

\[
Q(x) < 0,
\]

(12c)

and

\[
Q(x) - S(x)R^{-1}(x)S^T(x) < 0.
\]

(12d)

Finally, the following Lemma 3 will be used to handle the parametrical perturbation.

**Lemma 3 [2].** Given symmetric matrices \( \Omega \) and \( D, E \), of appropriate dimensions,

\[
\Omega + DF(t)E + E^TF^T(t)D^T < 0,
\]

(13a)

for all \( F(t) \) satisfying \( F^T(t)F(t) \leq I \), if and only if there exists some \( \varepsilon > 0 \) such that

\[
\Omega + \varepsilon DD^T + \varepsilon^{-1}E^TE < 0.
\]

(13b)

Firstly, we consider the nominal from system (8):

\[
\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - h(t)));
\]

(14)

For the nominal system (14), we will give a stability condition by using an integral inequality approach as follows.
Theorem 1: For given scalars \(h, \alpha,\) and \(h_d,\) system (14) is exponentially stable if there exist symmetry positive-definite matrices \(P = P^T > 0, Q = Q^T > 0, R = R^T > 0,\) \(U = U^T > 0, W = W^T > 0,\) diagonal matrices \(S \geq 0,\) \(\Lambda_1 \geq 0,\) \(\Lambda_2 \geq 0,\) and
\[
X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0 \quad \text{and} \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0
\]
such that the following LMIs hold:
\[
\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & 0 & \Omega_{16} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & 0 & 0 & \Omega_{26} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} & 0 & \Omega_{36} \\ \Omega_{14} & 0 & \Omega_{34}^T & \Omega_{44} & 0 & \\ 0 & 0 & 0 & \Omega_{45} & 0 & \\ \Omega_{16} & \Omega_{26} & \Omega_{36} & 0 & 0 & \Omega_{66} \end{bmatrix} \leq 0, \tag{15a}
\]
\[
R - X_{33} \geq 0, \tag{15b}
\]
\[
R - Y_{33} \geq 0. \tag{15c}
\]

where
\[
K = \text{diag}\{k_1, k_2, \ldots, k_n\}, \quad \Omega_{11} = -C^T P - PC + e^{2\alpha h}(Q + U) + e^{-2\alpha h}(hX_{11} + X_{13} + X_{13}^T),
\]
\[
\Omega_{12} = PA + 2\alpha S - C^T S + K \Lambda_2, \quad \Omega_{13} = PB, \quad \Omega_{14} = e^{-2\alpha h}(hX_{12} - X_{13}^T + X_{23}^T), \quad \Omega_{16} = -hC^TR,
\]
\[
\Omega_{22} = A^T S + SA + e^{2\alpha h}W - 2\Lambda_1, \quad \Omega_{23} = SB, \quad \Omega_{16} = hA^TR, \quad \Omega_{23} = -(1 - h_d)W - 2\Lambda_2,
\]
\[
\Omega_{34} = K \Lambda_2, \quad \Omega_{36} = hB^TR, \quad \Omega_{44} = -(1 - h_d)Q + e^{-2\alpha h}(hX_{22} - X_{23} - X_{23}^T + hY_{11} + Y_{13} + Y_{13}^T),
\]
\[
\Omega_{45} = e^{-2\alpha h}(hY_{12} - Y_{13} + Y_{23}), \Omega_{55} = -U + e^{-2\alpha h}(hY_{22} - Y_{23} - Y_{23}^T), \quad \Omega_{66} = hR.
\]

Proof: Choose the following Lyapunov-Krasovskii functional candidate to be
\[
V(t) = V_1(t) + V_2(t) + V_3(t), \tag{16}
\]
where
\[
V_1(t) = e^{2\alpha t}x^T(t)Px(t) + 2e^{2\alpha t} \int_0^t \sum_{i=1}^n s_i \int_0^{s_i(t)} x(s) ds,
\]
\[
V_2(t) = e^{2\alpha t} \int_{t-h(t)}^t e^{2\alpha s} [x^T(s)Qx(s) + g^T(x(s))Wg(x(s))] ds + e^{2\alpha t} \int_{-h(t)}^{t-h(t)} e^{2\alpha s} x^T(s)Ux(s) ds,
\]
\[
V_3(t) = \int_{-h(t)}^{t-h(t)} e^{2\alpha s} x^T(s)R\dot{x}(s) ds d\theta.
\]

Then, taking the time derivative of \(V(t)\) with respect to \(t\) along the system (14)
yield

\[ \dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t), \]  

(17)

First the derivative of \( V_1(t) \) is

\[
\dot{V}_1(x_t) = 2\alpha e^{2\alpha t} x^T(t) P x(t) + 2 e^{2\alpha t} x^T(t) P x(t) + 4\alpha e^{2\alpha t} g^T(x(t)) S x(t) + 2 e^{2\alpha t} g^T(x(t)) \dot{x}(t) \\
= e^{2\alpha t} [2\alpha x^T(t) P x(t) + 2 x^T(t) P x(t) + 4\alpha g^T(x(t)) S x(t) + 2 g^T(x(t)) \dot{x}(t)]. 
\]

(18)

Second, we get the bound of \( V_2(t) \) as

\[
\dot{V}_2(t) = e^{2\alpha t} \{ e^{2\alpha h} \lbrack x^T(t)(Q + U)x(t) + g^T(x(t)) W g(x(t)) \rbrack \\
- x^T(t-h) U x(t-h) - (1-\dot{h}(t)) \lbrack x^T(t-h(t)) Q x(t-h(t)) + g^T(x(t-h(t))) W g(x(t-h(t))) \rbrack \\
\leq e^{2\alpha t} \{ e^{2\alpha h} x^T(t)(Q + U)x(t) + g^T(x(t)) W g(x(t)) \rbrack - x^T(t-h) U x(t-h) \\
- (1-h(t)) \lbrack x^T(t-h(t)) Q x(t-h(t)) + g^T(x(t-h(t))) W g(x(t-h(t))) \rbrack. 
\]

(19)

Third, the bound of \( V_3(t) \) is as follows:

\[
\dot{V}_3(t) = \dot{h} e^{2\alpha t} x^T(t) \dot{x}(t) - \int_{t-h}^{t} \dot{h} e^{2\alpha s} x^T(s) \dot{x}(s) ds \\
\leq e^{2\alpha t} \dot{x}^T(t) h \dot{x}(t) - \int_{t-h}^{t} e^{2\alpha (t-h)} \dot{x}^T(s) \dot{x}(s) ds \\
= e^{2\alpha t} \dot{x}^T(t) h \dot{x}(t) - e^{2\alpha (t-h)} \int_{t-h}^{t} \dot{x}^T(s) \dot{x}(s) ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s) \dot{x}(s) ds \\
= e^{2\alpha t} \dot{x}^T(t) h \dot{x}(t) - e^{2\alpha (t-h)} \int_{t-h}^{t} \dot{x}^T(s) \dot{x}(s) ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s) \dot{x}(s) ds \\
- \int_{t-h}^{t-h(t)} \dot{x}^T(s) X_{33} \dot{x}(s) ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s) Y_{33} \dot{x}(s) ds \rbrack. 
\]

(20)

Using Lemma 2, the term \(-\int_{t-h(t)}^{t} \dot{x}^T(s) X_{33} \dot{x}(s) ds\) can be written that

\[
-\int_{t-h(t)}^{t} \dot{x}^T(s) X_{33} \dot{x}(s) ds \\
\leq \int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & \dot{x}^T(t-h(t)) \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\
X_{12} & X_{22} & X_{23} \\
X_{13} & X_{23} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\
x(t-h(t)) \\
\dot{x}(s) \end{bmatrix} ds \\
= x^T(t) \begin{bmatrix} h X_{11} + X_{13} + X_{13} \end{bmatrix} x(t) + x^T(t) \begin{bmatrix} h X_{12} - X_{13} + X_{13} \end{bmatrix} x(t-h(t)) \\
+ x^T(t-h(t)) \begin{bmatrix} h X_{12} - X_{13} + X_{23} \end{bmatrix} x(t) + x^T(t-h(t)) \begin{bmatrix} h X_{22} - X_{23} - X_{23} \end{bmatrix} x(t-h(t)). 
\]

(21)
Similarly, we have

\[-\int_{t-h}^{t} \dot{x}(s)Y_{33}\dot{x}(s)ds\]
\[\leq x^T(t-h)[hY_{11} + Y_{13}^T + Y_{13}]x(t-h(t)) + x^T(t-h)[hY_{12} - Y_{13} + Y_{23}^T]x(t-h)\]
\[+ x^T(t-h)[hY_{12} - Y_{13} + Y_{23}]x(t-h(t)) + x^T(t-h)[hY_{22} - Y_{23} - Y_{23}^T]x(t-h).\]  

(22)

The operator for term \( \dot{x}^T(t)hR\dot{x}(t) \) is as follows:

\[ \dot{x}^T(t)hR\dot{x}(t) \]
\[= [-Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))]^T(hR)[-Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))] \]
\[= x^T(t)hC^T R Cx(t) - x^T(t)hC^T R A g(x(t)) \]
\[= -x^T(t)hC^T R B g(x(t-h(t))) g^T(x(t))h A^T R Cx(t) \]
\[+ g^T(x(t))h A^T R A g(x(t)) + g^T(x(t))h A^T R B g(x(t-h(t))) \]
\[= -x^T(t)hA^T R x(t) + g^T(x(t-h(t)))h A^T R A g(x(t)) \]
\[+ g^T(x(t-h(t)))h A^T R B g(x(t-h(t))) \]  

(23)

From Eq.(9) for appropriately dimensioned diagonal matrices \( \Lambda, (i = 1, 2) \), we have

\[-2e^{2at}g^T(x(t))\Lambda \{g(x(t)) - Kx(t)\} \geq 0, \]  

(24)

and

\[-2e^{2at}g^T(x(t-h(t)))\Lambda \{g(x(t-h(t))) - Kx(t-h(t))\} \geq 0, \]  

(25)

Combining (18)-(25) yields

\[ \dot{V}(t) \leq e^{2at}\{\xi^T(t)\Xi(t) - e^{-2ah}\int_{t-h(t)}^{t} \dot{x}^T(s)(R - X_{33})\dot{x}(s)ds - \int_{t-h(t)}^{t} \dot{x}^T(s)(R - Y_{33})\dot{x}(s)ds\}, \]

(26)

\[\xi^T(t) = [x^T(t) \quad g^T(x(t)) \quad g^T(x(t-h(t))) \quad x^T(t-h(t)) \quad x^T(t-h)]\]
\[
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 \\
\Xi_{12} & \Xi_{22} & \Xi_{23} & 0 & 0 \\
\Xi_{13}^T & \Xi_{23}^T & \Xi_{33} & \Xi_{34} & 0 \\
\Xi_{14} & 0 & \Xi_{44} & \Xi_{45} & 0 \\
0 & 0 & 0 & \Xi_{55}^T & \Xi_{55}
\end{bmatrix}
\]

From Equation (14) and the Schur complement, it is easy to see that \( \dot{V}(x) < 0 \) holds if \( R - X \geq 0, R - Y \geq 0 \).

3. Exponential robust stability analysis

Based on Theorem 1, we have the following result for uncertain neural networks with time-varying delay (8).

**Theorem 2**: For given positive scalars \( h, \alpha, \) and \( h_d, \) the uncertain delayed neural networks with time-varying delay (8) is exponentially robust stable if there exist symmetric positive-definite matrices \( P = P^T > 0, \quad Q = Q^T > 0, \quad R = R^T > 0, \)

\( U = U^T > 0, \quad W = W^T > 0, \) diagonal matrices \( S \geq 0, \quad \Lambda_1 \geq 0, \quad \Lambda_2 \geq 0, \) a scalar \( \varepsilon > 0 \) and

\[
X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13}^T & X_{23}^T & X_{33}
\end{bmatrix} \geq 0,
\]

\[
Y = \begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} \\
Y_{12}^T & Y_{22} & Y_{23} \\
Y_{13}^T & Y_{23}^T & Y_{33}
\end{bmatrix} \geq 0
\]

such that the following LMIs are true

\[
\overline{\Omega} = \begin{bmatrix}
\Omega_{11} + \varepsilon E_c^T E_c & \Omega_{12} - \varepsilon E_c^T E_a & \Omega_{13} - \varepsilon E_c^T E_b & \Omega_{14} & 0 & \Omega_{16} & PD \\
\Omega_{12} - \varepsilon E_c^T E_c & \Omega_{22} + \varepsilon E_a^T E_a & \Omega_{23} + \varepsilon E_a^T E_b & 0 & 0 & \Omega_{26} & 0 \\
\Omega_{13}^T - \varepsilon E_c^T E_c & \Omega_{23} + \varepsilon E_a^T E_a & \Omega_{33} + \varepsilon E_a^T E_b & \Omega_{34} & 0 & \Omega_{36} & 0 \\
\Omega_{14} & 0 & \Omega_{24}^T & \Omega_{44} & \Omega_{45} & 0 & 0 \\
0 & 0 & 0 & \Omega_{35} & \Omega_{35} & 0 & 0 \\
0 & \Omega_{26} & \Omega_{26}^T & 0 & 0 & \Omega_{66} & hRD \\
0 & 0 & 0 & 0 & 0 & hD^T R & -\varepsilon I
\end{bmatrix} < 0, \quad (27a)
\]
and
\[ R - X_{33} \geq 0, \quad (27b) \]
\[ R - Y_{33} \geq 0, \quad (27c) \]

where \( \Omega_{ij}, (i, j = 1, \ldots, 6; i < j \leq 6) \) are defined in (15).

It is, incidentally, worth noting that the uncertain delayed neural networks with time-varying delay (8) is exponential stable, that is, the uncertain parts of the nominal system can be tolerated within allowable time delay \( h \) and exponential convergence rate \( \alpha \).

**Proof:** Replacing \( A, B, \) and \( C \) in (15) with \( A + DF(t)E_a, B + DF(t)E_b, \) and \( C + DF(t)E_c, \) respectively, we apply Lemma 2 for system (15) is equivalent to the following condition:

\[ \Omega + \Gamma_d F(t)\Gamma_e + \Gamma_d^T F(t)\Gamma_e < 0, \quad (28) \]

where \( \Gamma_d = \begin{bmatrix} PD & 0 & 0 & 0 & hRD \end{bmatrix}^T \) and \( \Gamma_e = \begin{bmatrix} -E_c & E_a & 0 & 0 & 0 \end{bmatrix} \).

According to Lemma 3, (28) is true if there exist a scalar \( \varepsilon > 0 \) such that the following inequality holds

\[ \Omega + \varepsilon^{-1} \Gamma_d^T \Gamma_d + \varepsilon \Gamma_e^T \Gamma_e < 0. \quad (29) \]

Applying the Schur complement shows that (29) is equivalent to (27a). This completes the proof.

If the upper bound of the derivative of time-varying delay \( h_d \) is unknown, Theorem 2 can be reduced to the result with \( Q = 0 \) and \( W = 0 \), we have the following Corollary 1.

**Corollary 1:** For given positive scalars \( h \) and \( \alpha \), the system (8) is exponentially robust stable if there exist symmetric positive-definite matrices \( P = P^T > 0, \)
\[ R = R^T > 0, \quad U = U^T > 0, \] diagonal matrices \( S \geq 0, \quad \Lambda_i \geq 0, \quad \Lambda_2 \geq 0, \quad \varepsilon > 0 \) and
\[ Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12} & Y_{22} & Y_{23} \\ Y_{13} & Y_{23} & Y_{33} \end{bmatrix} \geq 0 \] such that the following LMIs are true
\[ \tilde{\Omega} = \begin{bmatrix} \Omega_{11} + \varepsilon E_e^T E_e & \Omega_{12} - \varepsilon E_a^T E_e & \Omega_{13} - \varepsilon E_a^T E_b & \Omega_{14} & 0 & \Omega_{16} & 0 \\ \Omega_{12} - \varepsilon E_e^T E_e & \Omega_{22} + \varepsilon E_a^T E_a & \Omega_{23} + \varepsilon E_a^T E_b & 0 & 0 & \Omega_{26} & 0 \\ \Omega_{13} - \varepsilon E_e^T E_e & \Omega_{23} + \varepsilon E_b^T E_a & \Omega_{33} + \varepsilon E_b^T E_b & 0 & 0 & \Omega_{36} & 0 \\ \Omega_{14} & 0 & \Omega_{34} & \tilde{\Omega}_{44} & 0 & \Omega_{45} & 0 \\ 0 & 0 & 0 & \Omega_{55} & 0 & 0 & 0 \\ \Omega_{16} & \Omega_{26} & \Omega_{36} & 0 & 0 & \Omega_{66} & hRD \\ D^T P & 0 & 0 & 0 & 0 & hD^T R & -\varepsilon I \end{bmatrix} < 0, \quad (30a) \]

and

\[ \begin{align*}
R - X_{33} & \geq 0, \quad (30b) \\
R - Y_{33} & \geq 0, \quad (30c)
\end{align*} \]

where

\[ K = \text{diag}\{k_1, k_2, \ldots, k_n\}, \tilde{\Omega}_{11} = -C^T P - PC + \varepsilon^2 2aU + \varepsilon 2a(h X_{11} + X_{13} + X_{17}^T), \]
\[ \tilde{\Omega}_{22} = A^T S + SA - 2\Lambda_1, \tilde{\Omega}_{33} = -2\Lambda_2, \tilde{\Omega}_{44} = -2a(h X_{22} - X_{23} - X_{13}^T + hY_{11} + Y_{13} + Y_{17}). \]

**Proof:** If the matrix \( Q = W = 0 \) is selected in (15). This proof can be completed in a similar formulation to Theorem 1 and Theorem 2.

Based on that, a convex optimization problem is formulated to find the bound on the allowable delay time \( h \) and exponential convergence rate \( \alpha \) which maintains the recurrent neural network time delay with parameter uncertainties systems (8).

**Remark 1.** It is interesting to note that \( h \) and \( \alpha \) appear linearly in (15), (27) and (30). Thus a generalized eigenvalue problem (GEVP) as defined in Boyd, et al. [1] can be formulated to solve the minimum acceptable \( 1/h \) (or \( 1/\alpha \)) and therefore the maximum \( h \) (or \( \alpha \)) to maintain robust stability as judged by these conditions.

The lower bound of exponential convergence rate or the allowable time delay conditions can be determined by solving the following three optimization problems:

**Case 1:** to estimate the lower bound of exponential convergence rate \( \alpha > 0 \).

\[ \text{Op1 : } \begin{cases} \max \alpha \\ \text{s.t. condition (27) is satisfied, } h \text{ and } h_d \text{ fixed.} \end{cases} \]

**Case 2:** to estimate the allowable maximum time delay \( h \).

\[ \text{Op2 : } \begin{cases} \max h \\ \text{s.t. condition (27) is satisfied, } \alpha > 0 \text{ and } h_d \text{ fixed.} \end{cases} \]
Case 3: to estimate the allowable maximum change rate of time delay $h_d$.

\[ \text{Op3 : } \begin{cases} \max h_d \\ \text{s.t. condition (27) is satisfied, } \alpha > 0 \text{ and } h \text{ fixed.} \end{cases} \]

If the change rate of time delay is equal to 0, i.e., $h(t) = h$, then the systems (14) reduces to the neural networks with constant delay and consequently, Theorem 1 reduces to the following Corollary 1.

The lower bound of exponential convergence rate or the allowable time delay conditions can be determined by solving the following two optimization problems:

Case 4: to estimate the lower bound of exponential convergence rate $\alpha > 0$.

\[ \text{Op4 : } \begin{cases} \max \alpha \\ \text{s.t. condition (30) is satisfied, } h \text{ fixed.} \end{cases} \]

Case 5: to estimate the allowable maximum time delay $h$.

\[ \text{Op5 : } \begin{cases} \max h \\ \text{s.t. condition (30) is satisfied, } \alpha > 0 \text{ fixed.} \end{cases} \]

Remark 2. All the above optimization problems (Op1- Op5) can be solved by MATLAB LMI toolbox. Especially, Op1 and Op4 can estimate the lower bound of global exponential convergence rate $\alpha$, which means that the exponential convergence rate of any neural network included in (8) is at least equal to $\alpha$. It is useful in real-time optimal computation.

4. Numerical examples

This section provides four numerical examples to demonstrate the effectiveness of the presented criterion.

Example 1: Consider the delayed neural network (14) as follows:

\[
\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - h(t)))
\]

where

\[
C = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.
\]
The neuron activation functions are assumed to satisfy Assumption 1 with \( K = \text{diag}\{1,1\} \).

**Solution:** It is assumed that the upper bound \( \tilde{h} \) is fixed as 1. The exponential convergence rates for various \( h_d \)'s obtained from Theorem 1 and those in [5, 12, 15] are listed in Table 1. In the following Tables 1–2, “–” means that the results are not applicable to the corresponding cases, and “unknown \( h_d \)” means that \( h_d \) can be arbitrary values, even \( h_d \) is very large or \( h(t) \) is not differentiable.

On the other hand, if the exponential convergence rate of \( \alpha \) is fixed as 0.8, the upper bounds of \( \tilde{h} \) for various \( h_d \)'s from Theorem 1 and those in [5, 12, 15] are listed in Table 2.

From Table 1, it is clear that when the delay is time-invariant, i.e., \( h_d = 0 \), the obtained result in Theorem 1 is much better than that in [15]. Furthermore, when the delay is time-varying, the theorem in [15] fails to obtain the allowable exponential convergence rate of the exponential stable neural network system, but Theorem 1 in this paper can also obtain the significant better results than that in [5, 12], which guarantee the exponential stability of the neural networks. Moreover, when the exponential convergence rate of \( \alpha \) is fixed as 0.8, the upper bounds of \( \tilde{h} \) for various \( h_d \)'s derived by Theorem 1 are also better than those in [5, 12, 15] from Table 2. The reason is that, compared with [5, 12, 15], our results not only do not ignore any useful terms in the derivative of Lyapunov–Krasovskii functional but also consider the relationship among \( \tilde{h}, h(t) \), and \( h - h(t) \). Fig. 1 shows the state response of Example 1 with time delay \( \tilde{h} = 1.59 \), when the initial value is \([-1 \ 1]^T\).

**Example 2:** Consider the delayed neural network (14) as follows:

\[
\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))
\]  \hspace{1cm} (32)

where

\[
C = \begin{bmatrix}
4.1989 & 0 & 0 \\
0 & 0.7160 & 0 \\
0 & 0 & 1.9985 \\
\end{bmatrix}, \quad A = \begin{bmatrix}
0.4094 & 0.5719 & 0.2503 \\
1.0645 & 0.0410 & -0.9923 \\
-0.7439 & 0.6344 & 0.1066 \\
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0.3008 & 0 & 0 \\
0 & 0.3070 & 0 \\
0 & 0 & 0.3068
\end{bmatrix}.
\]

The neuron activation functions are assumed to satisfy Assumption 1 with 
\( K = \text{diag}\{0.4911, 0.9218, 0.6938\} \).

**Solution:** It is assumed that the exponential convergence rate of \( \alpha \) is fixed as zero. The upper bounds of \( h \) for various \( h_d \)'s from Theorem 1 and those in [5, 6, 7, 12] are listed in Table 3. It is also clear that the obtained upper bounds of \( \overline{h} \) in this correspondence paper are better than those in [5, 6, 7, 12], which guarantee the asymptotic stability of neural networks. The reason is that our results do not ignore any useful negative terms in the derivative of Lyapunov–Krasovskii functional compared with [5, 7], and our results also consider the relationship among \( \overline{h}, h(t) \), and \( h - h(t) \) compared with [6, 12].

**Example 3:** Consider the following uncertain delayed neural network:

\[
\dot{x}(t) = -Cx(t) + (A + \Delta A(t))g(x(t)) + (B + \Delta B(t))g(x(t - h(t))),
\]

where

\[
C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
A = \begin{bmatrix}
0.4 & 0.1 \\
0.1 & -0.5
\end{bmatrix},
B = \begin{bmatrix}
0.2 & 0.1 \\
0 & 0.2
\end{bmatrix},
D = \begin{bmatrix}
0.05 & 0 \\
0 & 0.05
\end{bmatrix},
E_a = E_b = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

The neuron activation functions are assumed to satisfy Assumption 1 with 
\( K = \text{diag}\{1, 1\} \).

**Solution:** We let \( h_d = 0.1 \) and \( h = 0.5 \) as [10] did, by Theorem 2, we can obtain the maximum allowable exponential convergence rate (MAECR) \( \bar{\alpha} \) size to be \( \bar{\alpha} = 0.4742 \). However, applying criteria in [10], the maximum value of \( \bar{\alpha} \) for the above system is 0.1. This example demonstrates that our robust stability condition gives a less conservative result. Hence, it is obvious that the results obtained from our simple method are less conservative than those obtained from the existing methods. The maximum allowable delay bound (MADB) \( \alpha \) for various \( h \) from Theorem 2 are listed in Table 4.
Example 4: Consider the following delayed neural network:

\[ \dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-h(t))) \]  \hspace{1cm} (34)\]

where

\[
C = \begin{bmatrix} 1.34 & 0 & 0 \\ 0 & 0.08 & 0 \\ 0 & 0 & 1.46 \end{bmatrix},
A = \begin{bmatrix} 1.08 & -0.52 & 0.05 \\ -0.08 & 0.68 & 0.19 \\ 0.14 & 0.29 & -0.58 \end{bmatrix},
B = \begin{bmatrix} 2.45 & -0.64 & 1.2 \\ 0.45 & 0.88 & 0.47 \\ 0.07 & -1.56 & 0.97 \end{bmatrix}.
\]

The neuron activation functions are assumed to satisfy Assumption 1 with \( K = \text{diag} \{ 0.31, 0.40, 0.23 \} \).

Solution: Let \( h_d \geq 1 \) and \( \alpha = 0.5 \). Using the Matlab LMIboolbox, the conditions of Theorem 2 can be met, then we get the global robust exponential stability in this example for \( h \leq 0.3970 \) and

\[
P = \begin{bmatrix} 0.1866 & -0.0648 & -0.0804 \\ -0.0648 & 1.0184 & 0.4300 \\ -0.0804 & 0.4300 & 1.1488 \end{bmatrix},
Q = \begin{bmatrix} 0.0027 & -0.0095 & -0.0188 \\ -0.0095 & 0.0367 & 0.0697 \\ -0.0188 & 0.0697 & 0.1418 \end{bmatrix},
R = \begin{bmatrix} 0.0559 & -0.0985 & 0.0115 \\ -0.0985 & 2.0362 & 0.1325 \\ 0.0115 & 0.1325 & 1.1883 \end{bmatrix},
W = \begin{bmatrix} 0.0046 & 0.0033 & -0.0397 \\ 0.0033 & 0.0155 & -0.0398 \\ -0.0397 & -0.0398 & 0.5184 \end{bmatrix},
U = \begin{bmatrix} 0.0559 & -0.0333 & -0.0421 \\ -0.0333 & 0.2848 & 0.1771 \\ -0.0421 & 0.1771 & 0.4583 \end{bmatrix},
X_{11} = \begin{bmatrix} 0.6600 & -0.0760 & -0.0218 \\ -0.0760 & 0.6376 & 0.0560 \\ -0.0218 & 0.0560 & 0.5618 \end{bmatrix},
X_{12} = \begin{bmatrix} -0.6176 & 0.0432 & -0.0831 \\ 0.0510 & -0.4650 & 0.0937 \\ -0.0121 & 0.0518 & -0.3444 \end{bmatrix},
X_{13} = \begin{bmatrix} -0.4185 & 0.1784 & -0.0811 \\ 0.0231 & -0.6490 & 0.0839 \\ 0.0230 & -0.0181 & -0.0991 \end{bmatrix},
X_{22} = \begin{bmatrix} 0.5958 & -0.0468 & 0.0380 \\ -0.0468 & 0.5207 & 0.0331 \\ 0.0380 & 0.0331 & 0.5891 \end{bmatrix},
X_{23} = \begin{bmatrix} 0.3502 & -0.1472 & 0.0416 \\ -0.0718 & 0.2440 & 0.0365 \\ 0.0860 & -0.1138 & 0.2952 \end{bmatrix},
X_{33} = \begin{bmatrix} 0.4916 & -0.0631 & 0.0484 \\ -0.0631 & 1.7896 & -0.0241 \\ 0.0484 & -0.0241 & 0.6812 \end{bmatrix},
Y_{11} = \begin{bmatrix} 0.0327 & -0.0196 & -0.0252 \\ -0.0196 & 0.1679 & 0.1037 \\ -0.0252 & 0.1037 & 0.2660 \end{bmatrix}.
\]
For this case, it can be verified that the stability conditions in [11, 14] are not applicable when $h_d \geq 1$. This implies that for this example the stability condition in Theorem 2 in this paper is less conservative than those in [11, 14].

5. Conclusion

In this paper, we have proposed some new delay dependent sufficient conditions for the global exponential robust stability analysis of a class of delayed neural networks with time-varying delays and parameter uncertainties. We have discussed the advantage of the assumption condition investigated in our paper over those in previous studies in the literature. Some global exponential stability criteria, which depend on time delay, are derived via the approach of the Lyapunov–Krasovskii functional. Four numerical examples demonstrate that the proposed method is an improvement over the existing ones.

References


Table 1. Allowable exponential convergence rate of $\alpha$ for various $h_d$ and $\bar{h} = 1$

<table>
<thead>
<tr>
<th>$h_d$</th>
<th>0</th>
<th>0.8</th>
<th>0.9</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>[15]</td>
<td>0.25</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>[5]</td>
<td>1.15</td>
<td>0.7538</td>
<td>0.6106</td>
<td>0.3391</td>
</tr>
<tr>
<td>[12]</td>
<td>1.15</td>
<td>0.8643</td>
<td>0.8344</td>
<td>0.8169</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>1.5997</td>
<td>1.1595</td>
<td>1.0526</td>
<td>0.9859</td>
</tr>
</tbody>
</table>

Table 2. Allowable upper bound of $\bar{h}$ for various $h_d$ and $\alpha = 0.8$

<table>
<thead>
<tr>
<th>$h_d$</th>
<th>0.8</th>
<th>0.9</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>[15]</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>[5]</td>
<td>1.2606</td>
<td>0.9442</td>
<td>0.8310</td>
</tr>
<tr>
<td>[12]</td>
<td>1.2787</td>
<td>1.0819</td>
<td>1.0366</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>2.4097</td>
<td>1.5995</td>
<td>1.1296</td>
</tr>
</tbody>
</table>

Table 3. Allowable upper bound of $\bar{h}$ for various $h_d$ and $\alpha = 0$

<table>
<thead>
<tr>
<th>$h_d$</th>
<th>0.8</th>
<th>0.9</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>[12]</td>
<td>12.252</td>
<td>6.441</td>
<td>5.831</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>21.1573</td>
<td>7.8999</td>
<td>7.8413</td>
</tr>
</tbody>
</table>

Table 4. Allowable exponential convergence rate of $\alpha$ for various $h$ and $h_d = 0.1$

<table>
<thead>
<tr>
<th>$h$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>2.1155</td>
<td>0.9885</td>
<td>0.4742</td>
<td>0.3069</td>
<td>0.2343</td>
<td>0.1270</td>
<td>0.0805</td>
</tr>
</tbody>
</table>
Figure 1. The simulation of the example 1 for $h = 1.59$ sec