具時變延遲類神經網路高木・菅野模糊型系統之穩定度分析-時延分解法

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摘 要

本論文旨在針對類神經網路高木・菅野模糊型時延系統之時延相關穩定化準則提出改善之方法。本文所提方法可處理解決時間延遲微分導數小於 1 之限制，使快速變動之時間延遲系統能獲得較大之延遲時間下確保時間延遲系統達到好漸近穩定條件。首先提出類神經網路高木・菅野模糊型時延系統之時延相關漸近穩定度之充分條件。基於線性矩陣不等式 (LMIs) 求解最大允許延遲時間的問題。採用 Lyapunov 泛函最佳化演算法設計迴授控制器推導出基於 LMI 的控制器設計方法。根據此充分條件，推導成一個凸優化問題，使用 LMI 工具箱求解器，可得到該系統的最大允許延遲時間 (MAUB)。文中舉例驗證與現有文獻結果相比較可得較寬廣的時間延遲範圍使得系統仍為漸近穩定。

關鍵字：模糊系統、類神經網路、線性矩陣不等式、積分不等式矩陣、最大允許延遲時間

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Stability Analysis of Takagi-Sugeno Fuzzy Systems for Neural Network with Time-Varying Delays- Delayed Decomposition Approach

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Abstract

In this paper, the stability of Takagi-Sugeno (T-S) fuzzy system for neural networks with time-varying delays is investigated. The constraint on the time-varying delay function is removed, which means that a fast time-varying delay is allowed. By developing a delay decomposition approach, the information of the delayed plant states can be taken into full consideration, and new delay-dependent sufficient stability criteria are obtained in terms of linear matrix inequalities (LMIs) which can be solved by various optimization algorithms. Numerical examples are included to show that the proposed method is effective and can provide less conservative results.

Keywords: Fuzzy system models; Neural networks; Linear matrix inequalities (LMIs); Integral inequality approach (IIA); Maximum admissible upper bound (MAUB)
1. Introduction

There has been a large amount of literature that studies the stability properties of linear time delay systems. Some fundamental results on these topics have been published in the literature [1-20] and the references therein. Recently, people have paid more and more attention on the robust stability of T-S fuzzy system with time delay [3] and time-varying delay [1, 2, 4, 7-14, 17, 18]. Unfortunately, the existing results always assume that the time-varying delay function is continuously differentiable and its derivative is smaller than one, see [10] for example, which is a rigorous constraint. Therefore, it is interesting but challenging to develop the robust stability condition without any constraint on the time-varying delay.

Recently, Fuzzy logic theory has shown to be an appealing and efficient approach to dealing with the analysis and synthesis problems for complex nonlinear systems. The well known T-S fuzzy model [16] is a popular and convenient tool in functional approximations. During the last decade, the problem of stability analysis and control synthesis for systems in T-S fuzzy model with time delay have been studied extensively and a lot of research works have been reported in the literature [2, 4, 5, 7, 10-14, 17, 18]. More recently, neural network (NN) have been extensively studied recurrently and involved in many different applications such as signal processing, optimization, fixed point computations, image processing, and other areas [8]. These applications are built upon the stability of the equilibrium of neural networks. Thus, the stability analysis is a necessary step for the design and applications of neural networks. For example, when a neural network is applied to solve the optimization problem, it must have a unique equilibrium which is globally stable. Therefore, stability analysis of neural networks has received much attention and various stability conditions have been obtained. It has been realized that significant time delays as a source of instability and bad performance may occur in neural processing and signal transmission. In contrast to pure neural networks or fuzzy systems, the fuzzy neural networks possess both their advantages. It combines the capability of fuzzy reasoning in handling uncertain information and the capability of artificial neural networks in learning from process. In the last decade, the concept of incorporating fuzzy logic into neural network has grown into a popular research topic [1, 8, 9, 15].

Stability criteria for T-S fuzzy systems are generally classified into two types: delay-dependent and delay-independent. Since delay-dependent criteria make use of
information on the lengths of delays, they are less conservative than delay-independent ones, especially when the delay is small. Based on a special Lyapunov functional approach and linear matrix inequality technology, improved delay-dependent stability criteria are derived by employing a delay partitioning approach [6, 19, 20] and free weighting matrix [13, 14]. While the proposed methods above need to decide more possible variables which may increase the complexity of the computation. However, there is room for further investigation. Useful terms tend to be ignored when the upper bound on the derivative of a Lyapunov-Krasovskii functional is estimated. Very recently, an integral inequality matrix approach (IIA) derived in [15] has been employed to derive some less conservative stability criteria. As a matter of fact, there exists conservativeness inevitably. This motivates us to establish a new delay-dependent condition and further reduce the conservatism.

However, as far as we know, in most existing literature, the above analyses have been treated separately. Up to now, the robust stability analysis for T-S fuzzy system for neural networks with time-varying delays hasn’t been fully studied, which is still open. In this paper, we extend the recent result [15] for T-S fuzzy system for neural networks with time-varying delays by employing an integral inequality approach (IIA). Under considering the relationship among the time-varying delay, its upper bound and their difference, some improved LMI-based stability criteria for T-S fuzzy system for neural networks with time-varying delays are obtained without ignoring any useful terms in the derivative of a Lyapunov functional. Finally, numerical examples are given to demonstrate the effectiveness and merits of the proposed method.

2. Main results

The neural network with time-varying delay in this paper can be described by the following normalized equations:

\[
\dot{u}(t) = -Cu(t) + Af(u(t)) + Bf(u(t - h(t))) + J, 
\]

where \( u(t) = [u_1(t), \ldots, u_n(t)]^T \in \mathbb{R}^n \) is the state vector with the \( n \) neurons; \( f(u(t)) = [f_1(u_1(t)), \ldots, f_n(u_n(t))]^T \in \mathbb{R}^n \) is called an activation function indicating how the \( j \)-th neuron responds to its input; \( C = \text{diag}(c_1, \ldots, c_n) \) is a diagonal matrix with each \( c_i > 0 \) controlling the rate with which the \( i \)-th unit will reset its potential to
the resting state in isolation when disconnected from the network and external inputs;  
\[ A = (a_j)_{n \times n} \]  and  \[ B = (b_j)_{n \times n} \]  are the feedback and the delayed feedback matrix, respectively;  \[ J = [J_1, \cdots, J_n]^T \in R^n \]  is a constant input vector, and  \( h(t) \)  is time-varying delay. We consider two different cases for time varying delays:

**Case I**:  \( h(t) \)  is a differentiable function, satisfying for all  \( t \geq 0 \):

\[
0 \leq h(t) \leq h \quad \text{and} \quad \dot{h}(t) \leq h_d,
\]

where  \( h \)  and  \( h_d \)  are some positive constants.

**Case II**:  \( h(t) \)  is not differentiable or the upper bound of the derivative of  \( h(t) \)  is unknown, and  \( h(t) \)  satisfies

\[
0 \leq h(t) \leq h,
\]

Throughout this paper, it is assumed that each of the activation functions  \( f_j(j = 1, 2, \ldots, n) \)  possess the following condition

\[
0 \leq \frac{f_i(u) - f_i(v)}{u - v} \leq k_i, \quad u \neq v \in R, i = 1, 2, \ldots, n,
\]

where  \( k_i(i = 1, 2, \ldots, n) \)  are positive constants.

Next, the equilibrium point  \( u^* = [u_1, \cdots, u_n]^T \)  of system (1) is shifted to the origin through the transformation  \( x(t) = u(t) - u^* \), then system (1) can be equivalently written as the following system

\[
\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t) - h(t)),
\]

where  \( x() = [x_1(), \cdots, x_n()]^T \),  \( g(x()) = [g_1(x_1()), \cdots, g_n(x_n())]^T \),

\[
g_i(x_i()) = f_i(x_i() + u_i()) - f_i(u_i^*), \quad i = 1, 2, \ldots, n.
\]
It is obvious that the function  \( g_j() (j = 1, 2, \ldots, n) \)  satisfies the following condition,
\[0 \leq \frac{g_i(x_i)}{x_i} \leq k, g_i(0) = 0, \forall x_i \neq 0, i = 1, 2, \ldots, n, \quad (6)\]

which is equivalent to

\[g_i(x_i)(g_i(x_i) - k|x_i|) \leq 0, g_i(0) = 0, \forall x_i \neq 0, i = 1, 2, \ldots, n. \quad (7)\]

The fuzzy model of (5) described by

**Plant Rule i**: If \(z_i(t)\) is \(M_{ij}\), and \(z_j(t)\) is \(M_{ip}\) then

\[
\begin{align*}
\dot{x}(t) &= -Cx(t) + A_{ig}(x(t)) + B_{ig}(x(t-h(t))), \\
x(t) &= \phi(t), t \in [-h, 0], i = 1, 2, \ldots, r,
\end{align*} \quad (8a)
\]

where \(z_i(t), z_2(t), \ldots, z_r(t)\) are the premise variables; \(M_{ij}, i = 1, 2, \ldots, r, j = 1, 2, \ldots, p\) are the fuzzy sets; \(x(t) \in R^n\) is the state vector; \(\phi(t)\) is a vector-valued initial condition; the scalars \(r\) is the number of IF–Then rules.

By fuzzy blending, the overall fuzzy model is inferred as follows:

\[
\begin{align*}
\dot{x}(t) &= \frac{\sum_{i=1}^{r} w_i(z(t))[-C x(t) + A_{ig}(x(t)) + B_{ig}(x(t-h(t)))]}{\sum_{i=1}^{r} w_i(z(t))} \\
&= \sum_{i=1}^{r} \mu_i(z(t))[-C x(t) + A_{ig}(x(t)) + B_{ig}(x(t-h(t)))] \\
&= -C x(t) + A g(x(t)) + B g(x(t-h(t))), \\
x(t) &= \phi(t), t \in [-h, 0],
\end{align*} \quad (9a)
\]

where \(z = [z_1, z_2, \ldots, z_p]; w_i: R^p \rightarrow [0, 1], i = 1, 2, \ldots, r,\) is the membership function of the system with respect to the plant rule \(i; \theta_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}; C = \sum_{i=1}^{r} \theta_i(z(t)) C_i; A = \sum_{i=1}^{r} \theta_i(z(t)) A_i, \) and \(B = \sum_{i=1}^{r} \theta_i(z(t)) B_i.\) It is assumed that \(w_i(z(t)) \geq 0, i = 1, 2, \ldots, r,\)

\[\sum_{i=1}^{r} w_i(z(t)) \geq 0 \forall t, \text{ so we have } \theta_i(z(t)) \geq 0, \sum_{i=1}^{r} \theta_i(z(t)) = 1.\]

In the following, we will develop some practically computable stability criteria for the system described(9). The following lemmas are useful in deriving the criteria.
Lemma 1 [15]. For any positive semi-definite matrices

\[
X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13}^T & X_{23}^T & X_{33}
\end{bmatrix} \geq 0,
\]

(10a)

the following integral inequality holds:

\[
-\int_{t-h(t)}^{t} \dot{x}^T(s)X_{33}\dot{x}(s)ds \leq
\int_{t-h(t)}^{t} \begin{bmatrix}
x^T(t)
\dot{x}^T(t-h(t))
\dot{x}^T(s)
\end{bmatrix}
\begin{bmatrix}
X_{11} & X_{12} & X_{13}
X_{12}^T & X_{22} & X_{23}
X_{13}^T & X_{23}^T & X_{33}
\end{bmatrix}
\begin{bmatrix}
x(t)
x(t-h(t))
\dot{x}(s)
\end{bmatrix}ds.
\]

(10b)

Second, we introduce the following Schur complement, essential to proving our results.

Lemma 2 [3]. The following matrix inequality

\[
\begin{bmatrix}
Q(x) & S(x) \\
S^T(x) & R(x)
\end{bmatrix} < 0,
\]

(11a)

where \( Q(x) = Q^T(x), R(x) = R^T(x) \) and \( S(x) \) depend on affine on \( x \), is equivalent to

\[
R(x) < 0,
\]

(11b)

\[
Q(x) < 0,
\]

(11c)

and

\[
Q(x) - S(x)R^{-1}(x)S^T(x) < 0.
\]

(11d)

This paper finds new stability criteria less conservative than the existing results. For the system (1)–(9), we give stability condition by using delayed decomposition approach as follows.

Theorem 1: In Case I, if \( 0 \leq h(t) \leq \alpha h \), for given three scalars \( h, \alpha, \) and \( h_d \). Then, for any delay \( h(t) \) satisfy \( 0 \leq h(t) \leq h, \ h(t) \leq h_d \), and \( 0 < \alpha < 1 \), the system described by (9) with (2) is asymptotically stable if there exist matrices \( P = P^T > 0, U > 0 \),
$Q_j = Q_j^T > 0, \quad R_j = R_j^T > 0, (j=1,2,3),$ diagonal matrices $S \geq 0, \quad A_1 \geq 0, A_2 \geq 0,$ and

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23} & X_{33} \end{bmatrix} \geq 0, \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23} & Y_{33} \end{bmatrix} \geq 0, \quad Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13}^T & Z_{23} & Z_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold for $i \in \{1,2,\ldots,r\}$:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & 0 & 0 & \Omega_{47} & \Omega_{18} & \Omega_{49} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & 0 & 0 & 0 & \Omega_{27} & \Omega_{28} & \Omega_{29} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & 0 & 0 & \Omega_{37} & \Omega_{38} & \Omega_{39} \\ \Omega_{14} & 0 & \Omega_{14}^T & \Omega_{44} & \Omega_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_{45}^T & \Omega_{55} & \Omega_{56} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_{56}^T & \Omega_{66} & 0 & 0 & 0 \\ \Omega_{17}^T & \Omega_{27}^T & \Omega_{27} & 0 & 0 & \Omega_{77} & 0 & 0 \\ \Omega_{18}^T & \Omega_{28}^T & \Omega_{28} & 0 & 0 & 0 & \Omega_{88} & 0 \\ \Omega_{19}^T & \Omega_{29}^T & \Omega_{29} & 0 & 0 & 0 & 0 & \Omega_{99} \end{bmatrix} < 0, \quad (12a)$$

and

$$R_1 - X_{33} \geq 0, R_2 - Y_{33} \geq 0, R_1 + (1-h_d)R_3 - Z_{33} \geq 0, \quad (12b)$$

where

$$K = \text{diag}\{k_1, k_2, \ldots, k_n\}, \quad \Omega_{11} = -C_i^T P - PC_i + Q_1 + Q_3 + \alpha h Z_{11} + Z_{13} + Z_{11}^T,$$

$$\Omega_{12} = P A_i - C_i^T S + K A_1, \quad \Omega_{13} = P B_i, \quad \Omega_{14} = \alpha h Z_{12} - Z_{13} + Z_{23}^T, \quad \Omega_{17} = -\alpha C_i^T R_1,$$

$$\Omega_{18} = -(1-\alpha)h A_i^T R_2, \quad \Omega_{19} = -\alpha C_i^T R_3, \quad \Omega_{22} = A_i^T S + S A_i + U - 2A_1, \quad \Omega_{23} = S B_i,$$

$$\Omega_{27} = \alpha h A_i^T R_1, \quad \Omega_{28} = -(1-\alpha)h A_i^T R_2, \quad \Omega_{29} = \alpha h A_i^T R_3, \quad \Omega_{33} = -(1-h_d)U - 2A_2,$$

$$\Omega_{34} = K A_2, \quad \Omega_{37} = \alpha h B_i^T R_1, \quad \Omega_{38} = (1-\alpha)h B_i^T R_2, \quad \Omega_{39} = \alpha h B_i^T R_3,$$

$$\Omega_{44} = -(1-h_d)Q_3 + \alpha h X_{11} + X_{13} + X_{13}^T + \alpha h Z_{22} - Z_{23} - Z_{23}^T, \quad \Omega_{45} = \alpha h X_{12} - X_{13} + X_{13}^T,$$

$$\Omega_{55} = Q_2 - Q_3 + \alpha h X_{22} - X_{23} - X_{23}^T + (1-\alpha)h Y_{11} + Y_{13} + Y_{13}^T, \quad \Omega_{56} = (1-\alpha)h Y_{12} - Y_{13} + Y_{13}^T,$$

$$\Omega_{66} = -Q_2 + (1-\alpha)h Y_{22} - Y_{23} - Y_{23}^T, \quad \Omega_{77} = -\alpha h R_1, \quad \Omega_{88} = (1-\alpha)h R_2, \quad \Omega_{99} = -\alpha h R_3.$$

**Proof:** See the Appendix A for a sketch of the proof.

**Theorem 2:** In Case I, if $\alpha h \leq h(t) \leq h$, for given three scalars $h, \alpha,$ and $h_d$. Then, for any delay $h(t)$ satisfy $0 \leq h(t) \leq h$, $\dot{h}(t) \leq h_d$, and $0 < \alpha < 1$, the system described by (9) with (2) is asymptotically stable if there exist matrices $P = P^T > 0,$ $U > 0, Q_j = Q_j^T > 0, \quad R_j = R_j^T \geq 0, (j=1,2,3),$ diagonal matrices $S \geq 0, \quad A_1 \geq 0, A_2 \geq 0,$
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and \( X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13} & X_{23}^T & X_{33} \end{bmatrix} \geq 0 \), \( Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13} & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0 \), \( Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13} & Z_{23}^T & Z_{33} \end{bmatrix} \geq 0 \)

such that the following LMIs hold for \( i \in \{1, 2, \ldots, r\} : \)

\[
\begin{bmatrix}
\Omega_{i1} & \Omega_{i2} & \Omega_{i3} & 0 & 0 & \Omega_{i5} & 0 & \Omega_{i7} & \Omega_{i8} & \Omega_{i9} \\
\Omega_{i2}^T & \Omega_{i22} & \Omega_{i23} & 0 & 0 & 0 & \Omega_{i27} & \Omega_{i28} & \Omega_{i29} \\
\Omega_{i3}^T & \Omega_{i32} & \Omega_{i33} & \Omega_{i34} & 0 & 0 & \Omega_{i27} & \Omega_{i38} & \Omega_{i29} \\
0 & 0 & \Omega_{i44} & \Omega_{i45} & \Omega_{i46} & 0 & 0 & 0 & 0 \\
0 & 0 & \Omega_{i46} & 0 & \Omega_{i46} & 0 & 0 & 0 & 0 \\
\Omega_{i5}^T & \Omega_{i52} & \Omega_{i53} & \Omega_{i54} & \Omega_{i55} & 0 & 0 & 0 & 0 \\
\Omega_{i6}^T & \Omega_{i62} & \Omega_{i63} & \Omega_{i64} & \Omega_{i65} & 0 & 0 & 0 & 0 \\
\Omega_{i7}^T & \Omega_{i72} & \Omega_{i73} & \Omega_{i74} & \Omega_{i75} & 0 & 0 & 0 & 0 \\
\Omega_{i8}^T & \Omega_{i82} & \Omega_{i83} & \Omega_{i84} & \Omega_{i85} & 0 & 0 & 0 & 0 \\
\Omega_{i9}^T & \Omega_{i92} & \Omega_{i93} & \Omega_{i94} & \Omega_{i95} & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0, \quad (13a)
\]

and

\[
R_1 + (1 - h_d)R_3 - X_{33} \geq 0, \quad R_2 + (1 - h_d)R_3 - Y_{33} \geq 0, \quad R_2 - Z_{33} \geq 0, \quad (13b)
\]

where

\[
\begin{align*}
\Omega_{i11} &= -C_i^T P - PC_i + Q_i + Q_2 + \alpha h X_{11} + X_{13} + X_{13}^T, \\
\Omega_{i15} &= \alpha h X_{12} - X_{13} + X_{13}^T, \\
\Omega_{i18} &= -(1 - \alpha) h C_i R_2, \\
\Omega_{i19} &= -h C_i^T R_3, \\
\Omega_{i22} &= A_i^T S + A_i U - 2 \Lambda_1, \\
\Omega_{i27} &= \alpha h A_i^T R_1, \\
\Omega_{i28} &= (1 - \alpha) h A_i^T R_2, \\
\Omega_{i29} &= h B_i R_3, \\
\Omega_{i34} &= -(1 - \alpha) h Y_{22} - Y_{23} - Y_{23}^T, \\
\Omega_{i35} &= (1 - \alpha) h Y_{12} - Y_{13} + Y_{13}^T, \\
\Omega_{i38} &= -(1 - \alpha) h Z_{11} + Z_{13} + Z_{13}^T, \\
\Omega_{i39} &= (1 - \alpha) h Z_{22} - Z_{23} - Z_{23}^T, \\
\Omega_{i46} &= -(1 - \alpha) h R_1, \\
\Omega_{i48} &= (1 - \alpha) h R_2, \\
\Omega_{i49} &= h R_3,
\end{align*}
\]

\[
K = \text{diag}\{k_1, k_2, \ldots, k_a\}.
\]

**Proof:** See the Appendix B.

**Theorem 3:** In Case II, if \( 0 \leq h(t) \leq \alpha h \), for given two scalars \( h \) and \( \alpha \). Then, for any delay \( h(t) \) satisfy \( 0 \leq h(t) \leq h \) and \( 0 < \alpha < 1 \), the system described by (9) with (3) is asymptotically stable if there exist matrices \( P = P^T > 0 \), \( Q_j = Q_j^T > 0 \), \( R_j = R_j^T > 0 \), \( j = 1, 2 \), diagonal matrices \( S \geq 0 \), \( \Lambda_1 \geq 0 \), \( \Lambda_2 \geq 0 \), and
such that the following LMIs hold for $i \in \{1, 2, \ldots, r\}$:

$$
\Psi = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & 0 & 0 & \Psi_{17} & \Psi_{18} \\
\Psi_{12}^T & 0 & 0 & 0 & \Psi_{27} & \Psi_{28} \\
\Psi_{13}^T & \Psi_{23} & \Psi_{33} & \Psi_{34} & 0 & 0 & \Psi_{37} & \Psi_{38} \\
\Psi_{14}^T & 0 & 0 & 0 & \Psi_{45} & \Psi_{55} & \Psi_{56} & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_{56} & \Psi_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_{77} & 0 \\
\Psi_{17}^T & \Psi_{27} & \Psi_{37} & 0 & 0 & 0 \\
\Psi_{18}^T & \Psi_{28} & \Psi_{38} & 0 & 0 & 0 & 0 & \Psi_{88}
\end{bmatrix} < 0, \quad (14a)
$$

(14b)

$$
R_1 - X_{33} \geq 0, R_2 - Y_{33} \geq 0, R_1 - Z_{33} \geq 0,
$$

where

$$
\Psi_{11} = -C_i^T P - P C_i + Q_i + Q_3 + \alpha h Z_{11} + Z_{13} + Z_{13}^T, \Psi_{12} = P A_i - C_i^T S + K_1 \Lambda_1, \Psi_{13} = P B_i,
$$

$$
\Psi_{14} = \alpha h Z_{12} - Z_{13} + Z_{13}^T, \Psi_{17} = -\alpha h C_i^T R_3, \Psi_{18} = -(1 - \alpha) h C_i^T R_2,
$$

$$
\Psi_{22} = A_i^T S + S A_i + U - 2 \Lambda_1, \Psi_{23} = S B_i, \Psi_{27} = \alpha h A_i^T R_1, \Psi_{28} = (1 - \alpha) h A_i^T R_2,
$$

$$
\Psi_{33} = -(1 - h_d) U - 2 \Lambda_2, \Psi_{34} = K \Lambda_2, \Psi_{37} = \alpha h B_i^T R_1, \Psi_{38} = (1 - \alpha) h B_i^T R_2,
$$

$$
\Psi_{44} = -(1 - h_d) Q_3 + \alpha h X_{11} + X_{13} + X_{13}^T + \alpha h Z_{22} - Z_{23} - Z_{23}^T, \Psi_{45} = \alpha h X_{12} - X_{13} + X_{13}^T,
$$

$$
\Psi_{55} = Q_2 - Q_i + \alpha h X_{22} - X_{23} - X_{23}^T + (1 - \alpha) h Y_{11} + Y_{13} + Y_{13}^T,
$$

$$
\Psi_{56} = (1 - \alpha) h Y_{12} - Y_{13} + Y_{13}^T, \Psi_{56} = Q_2 + (1 - \alpha) h Y_{22} - Y_{23} - Y_{23}^T,
$$

$$
\Psi_{77} = -\alpha h R_1, \Psi_{88} = (1 - \alpha) h R_2, K = diag\{k_1, k_2, \ldots, k_n\}.
$$

**Theorem 4:** In Case II, if $\alpha h \leq h(t) \leq h$, for given two scalars $h$ and $\alpha$. Then, for any delay $h(t)$ satisfy $0 \leq h(t) \leq h$ and $0 < \alpha < 1$, the system described by (9) with (3) is asymptotically stable if there exist matrices $P = P^T > 0$, $Q_j = Q_j^T > 0$, $R_j = R_j^T > 0 (j = 1, 2)$, diagonal matrices $S \geq 0$, $\Lambda_i \geq 0, \Lambda_2 \geq 0$, and

$$
X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13}^T & X_{23} & X_{33}
\end{bmatrix} \geq 0, \quad Y = \begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} \\
Y_{12}^T & Y_{22} & Y_{23} \\
Y_{13}^T & Y_{23} & Y_{33}
\end{bmatrix} \geq 0, \quad Z = \begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{12}^T & Z_{22} & Z_{23} \\
Z_{13}^T & Z_{23} & Z_{33}
\end{bmatrix} \geq 0
$$

such that the following LMIs hold for $i \in \{1, 2, \ldots, r\}$:
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\[
\overline{\Psi} = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & 0 & \Psi_{15} & 0 & \Psi_{17} & \Psi_{18} \\
\Psi_{12}^T & \Psi_{22} & \Psi_{23} & 0 & 0 & 0 & \Psi_{27} & \Psi_{28} \\
\Psi_{13}^T & \Psi_{23} & \Psi_{33} & \Psi_{34} & 0 & 0 & \Psi_{37} & \Psi_{38} \\
0 & 0 & \Psi_{34}^T & \Psi_{44} & \Psi_{45} & \Psi_{46} & 0 & 0 \\
0 & 0 & 0 & \Psi_{46}^T & \Psi_{55} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_{56}^T & 0 & \Psi_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & \Psi_{77}^T & 0 \\
\Psi_{18}^T & \Psi_{28}^T & \Psi_{38}^T & 0 & 0 & 0 & 0 & \Psi_{88}^T
\end{bmatrix} < 0, \tag{15a}
\]

where

\[
\begin{align*}
\Psi_{11} &= -C_i^T P - P C_i + Q_i + \alpha h X_{11} + X_{13} + X_{13}^T, \\
\Psi_{13} &= P A_i + C_i^T S + K A_i, \\
\Psi_{15} &= \alpha h X_{12} - X_{13} + X_{13}^T, \\
\Psi_{17} &= \alpha h C_i^T R_i, \\
\Psi_{18} &= -(1 - \alpha) h C_i^T R_i, \\
\Psi_{22} &= A_i^T S + S A_i + U - 2 \Lambda_t, \\
\Psi_{23} &= S B_i, \\
\Psi_{27} &= \alpha h A_i^T R_i, \\
\Psi_{28} &= -(1 - \alpha) h A_i^T R_i, \\
\Psi_{33} &= -(1 - \alpha) h U - 2 \Lambda_2, \\
\Psi_{34} &= K A_2, \\
\Psi_{37} &= \alpha h B_i^T R_i, \\
\Psi_{38} &= -(1 - \alpha) h B_i^T R_i, \\
\Psi_{44} &= (1 - \alpha) h Y_{22} - Y_{23} - Y_{23}^T, \\
\Psi_{45} &= (1 - \alpha) h Z_{12} - Z_{13} + Z_{13}^T, \\
\Psi_{46} &= -Q_2 + (1 - \alpha) h Z_{22} - Z_{23} - Z_{23}^T, \\
\Psi_{55} &= Q_2 - Q_4 + \alpha h X_{22} - X_{23} - X_{23}^T, \\
\Psi_{66} &= -Q_2 + (1 - \alpha) h Y_{11} + Y_{13} + Y_{13}^T, \\
\Psi_{77} &= -\alpha h R_i, \\
\Psi_{88} &= -(1 - \alpha) h R_i.
\end{align*}
\]

The matrix \( K \) is defined as \( K = \text{diag} \{ k_1, k_2, \ldots, k_n \} \).

**Proof:** In Case II for Theorems 3 and 4, a Lyapunov functional can be chosen as (A1) with \( Q_3 = R_3 = U = 0 \). Similar to the above analysis, one can get that \( \dot{V}(x_i) < 0 \) holds if \( \Psi < 0(\overline{\Psi} < 0) \). Thus, the proof is completed.

**Remark 1:** In the proof of Theorems 1-4, the interval \([t - h, t]\) is divided into subintervals \([t - h, t - \alpha h]\) and \([t - \alpha h, t]\), information of delayed state \( x(t - \alpha h) \) can be taken into account. It is clear that the Lyapunov function defined in Theorems 1-4 is more general than the ones in \([1, 2, 4, 5]\), etc.

**Remark 2:** In the previous works except \([1, 2, 4, 5]\), the time delay term \( h(t) \) was usually estimated as \( h \) when estimating the upper bound of some cross term, this may lead to increasing conservatism inevitably. In Theorems 1-4, the value of the
upper bound of some cross term is estimated more exactly than the previous methods since $h(t)$ is confined to the subintervals $0 \leq h(t) \leq \alpha h$ or $\alpha h \leq h(t) \leq h$. So, such decomposition method may lead to reduction of conservatism.

**Remark 3:** In the stability problem, maximum admissible upper bound (MAUB) $\bar{h}$ that ensures neural network with time-varying delay system (9) is stabilizable for any $\bar{h}$ can be determined by solving the following quasi-convex optimization problem when the other bound of time-varying delay $\bar{h}$ is known.

$$
\begin{align*}
\text{Maximize} & \quad \bar{h} \\
\text{Subject to} & \quad (12) ((13),(14),(15))
\end{align*} \quad (16)
$$

Inequality (16) is a convex optimization problem and can be obtained efficiently using the MATLAB LMI Toolbox.

Seeking an appropriate $\alpha$ satisfying $0 < \alpha < 1$, such that upper bound $h$ of delay $0 \leq h(t) \leq \alpha h$, subjecting to (12) is maximal, we give an algorithm as follows:

**Algorithm 1. (Maximizing $h > 0$)**

Step 1: For given $h_0$, choose an upper bound on $h$ satisfying (12), and then select this upper bound as the initial value $h_0$ of $h$.

Step 2: Set appropriate step lengths, $h_{\text{step}}$ and $\alpha_{\text{step}}$, for $h$ and $\alpha$, respectively. Set $k$ as a counter, and choose $k = 1$. Meanwhile, let $h = h_0 + h_{\text{step}}$ and initial value $\alpha_0$ of $\alpha$ equals to $\alpha_{\text{step}}$.

Step 3: Let $\alpha = k\alpha_{\text{step}}$ if inequality (9) is feasible, go to Step 4; otherwise, go to step 5.

Step 4: Let $h_0 = h, \alpha_0 = \alpha, k = 1$ and $h = h_0 + h_{\text{step}}$, go to Step 3.

Step 5: Let $k = k + 1$. If $k\alpha_{\text{step}} < 1$, then go to Step 3; otherwise, stop.

**Remark 4.** For Algorithm 1, final $h_0$ is the desired maximum of the upper bound of delay $h(t)$ satisfying (12) and $\alpha_0$ is corresponding value of $\alpha$.

**Remark 5.** Similar to Algorithm 1, we also find an appropriate scalar $\delta$, such that the upper bound of delay $0 \leq h(t) \leq \delta h$, for any $h_0$, subjecting to (13) attains maximum.

**Remark 6.** Similar to Algorithm 1, an algorithm seeking an appropriate $\alpha$, such that upper bound of delay $\alpha h \leq h(t) \leq h$, subjecting to (14) is maximal is easily obtained.

**Remark 7.** Similar to Algorithm 1, an algorithm seeking appropriate $\alpha$, such that upper bound of delay $\alpha h \leq h(t) \leq h$, for any $h_0$, subjecting (15) is maximal can be
3. Illustrative examples

In this section, we use two examples to illustrate the effectiveness and merits of our results. Example 1 is selected from [4, 5] with some modifications in order to show that our delay-dependent result in Theorem 1 works better than the result given in [4, 5]. Example 2 is used to show that our delay-dependent result in Theorem 1 is feasible for the design of fuzzy controller while the existing delay-dependent results in [1, 2] do not. It can be shown that the delay-dependent stability condition in this paper is the best performance.

**Example 1.** Consider a time delayed fuzzy system. The T–S fuzzy model of this fuzzy neural network system is of the following form:

Plant rules:
Rule 1: If \( x(t) \) is \( M_1 \), then

\[
\dot{x}(t) = -C_1 x(t) + A_1 g(x(t)) + B_1 g(x(t-h(t))), \tag{17a}
\]

Rule 2: If \( x(t) \) is \( M_2 \), then

\[
\dot{x}(t) = -C_2 x(t) + A_2 g(x(t)) + B_2 g(x(t-h(t))), \tag{17b}
\]

and the membership function for rule 1 and rule 2 are

\[
M_1(x(t)) = \frac{1}{1 + \exp(-2x(t))}, M_2(x(t)) = 1 - M_1(x(t)),
\]

where

\[
C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.3 & -0.4 \\ 0.28 & 0.7 \end{bmatrix}, B_1 = \begin{bmatrix} -0.2 & 0.2 \\ -0.3 & 0.12 \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -0.5 & -0.26 \\ 0.28 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 & -0.02 \\ -0.22 & 0.1 \end{bmatrix}.
\]
The neuron activation functions are assumed to satisfy Assumption 1 with $K = \text{diag}[1, 1]$.

**Solution:** For $h_d = 1.5(\alpha = 0.5)$, by Theorem 1, we can obtain the maximum upper bound on the allowable size to be $\bar{h} = 6.0398$. However, applying criteria in [4, 5], the maximum value of $h$ for the above system is 3.2913 and 1.5035. It is seen that our results improve the existing results [4, 5]. In case of $\bar{h} = 6.0398$, solving Theorem 1 yields the following set of feasible solutions

$$
\begin{align*}
P &= \begin{bmatrix} 41.5075 & -4.7125 \\ -4.7125 & 19.8944 \end{bmatrix},
Q_1 &= \begin{bmatrix} 0.0313 & -0.0062 \\ -0.0062 & 0.0077 \end{bmatrix},
Q_2 &= \begin{bmatrix} 0.0311 & -0.0062 \\ -0.0062 & 0.0077 \end{bmatrix},
R_1 &= \begin{bmatrix} 0.3636 & -0.0728 \\ -0.0728 & 0.0921 \end{bmatrix},
R_2 &= \begin{bmatrix} 13.0620 & -3.4227 \\ -3.4227 & 7.8291 \end{bmatrix},
R_3 &= \begin{bmatrix} 0.0098 & -0.0025 \\ -0.0025 & 0.0079 \end{bmatrix},
X_{11} &= \begin{bmatrix} 0.0399 & -0.0080 \\ -0.0080 & 0.0101 \end{bmatrix},
X_{12} &= \begin{bmatrix} -0.0381 & 0.0074 \\ 0.0074 & -0.0095 \end{bmatrix},
X_{13} &= \begin{bmatrix} -0.1193 & 0.0238 \\ 0.0238 & -0.0295 \end{bmatrix},
X_{22} &= \begin{bmatrix} 0.3810 & -0.0991 \\ -0.0991 & 0.2095 \end{bmatrix},
X_{23} &= \begin{bmatrix} 0.1152 & -0.0223 \\ -0.0223 & 0.0288 \end{bmatrix},
X_{33} &= \begin{bmatrix} 0.3602 & -0.0719 \\ -0.0719 & 0.0892 \end{bmatrix},
Y_{11} &= \begin{bmatrix} 1.5064 & -0.3826 \\ -0.3826 & 0.9199 \end{bmatrix},
Y_{12} &= \begin{bmatrix} -1.1453 & 0.2838 \\ 0.2838 & -0.7132 \end{bmatrix},
Y_{13} &= \begin{bmatrix} -3.6347 & 0.9262 \\ 0.9262 & -2.2230 \end{bmatrix},
Y_{22} &= \begin{bmatrix} 1.6091 & -0.4097 \\ -0.4097 & 0.9832 \end{bmatrix},
Y_{23} &= \begin{bmatrix} 3.5816 & -0.9046 \\ -0.9046 & 2.2029 \end{bmatrix},
Y_{33} &= \begin{bmatrix} 11.3676 & -2.9511 \\ -2.9511 & 6.8674 \end{bmatrix},
Z_{11} &= \begin{bmatrix} 1.4394 & -0.3764 \\ -0.3764 & 0.8602 \end{bmatrix},
Z_{12} &= \begin{bmatrix} -1.4304 & 0.3746 \\ 0.3746 & -0.8579 \end{bmatrix},
Z_{13} &= \begin{bmatrix} -4.3197 & 1.1315 \\ 1.1315 & -2.5908 \end{bmatrix},
Z_{22} &= \begin{bmatrix} 1.4322 & -0.3753 \\ -0.3753 & 0.8585 \end{bmatrix},
Z_{23} &= \begin{bmatrix} 4.3241 & -1.1330 \\ -1.1330 & 2.5915 \end{bmatrix},
Z_{33} &= \begin{bmatrix} 13.0587 & -3.4218 \\ -3.4218 & 7.8265 \end{bmatrix},
U &= \begin{bmatrix} 0.0023 & 0.0006 \\ 0.0006 & 0.0018 \end{bmatrix},
A_1 &= \begin{bmatrix} 32.2313 & 0 \\ 0 & 23.6533 \end{bmatrix},
A_2 &= \begin{bmatrix} 4.4198 & 0 \\ 0 & 2.7764 \end{bmatrix}.
\end{align*}
$$

Therefore, the fuzzy neural networks with time-varying delays (17) are globally asymptotically stable.
**Example 2.** Consider a time delayed fuzzy system. The T–S fuzzy model of this fuzzy system is of the following form:

Plant rules:

Rule 1: If \( x_1(t) \) is \( M_1 \), then
\[
\dot{x}(t) = -C_1 x(t) + A_1 g(x(t)) + B_1 g(x(t - h(t)))
\]  
(18a)

Rule 2: If \( x_1(t) \) is \( M_2 \), then
\[
\dot{x}(t) = -C_2 x(t) + A_2 g(x(t)) + B_2 g(x(t - h(t)))
\]  
(18b)

and the membership function for rule 1 and rule 2 are

\[
M_1(x_1(t)) = \frac{1}{1 + \exp(-2x_1(t))}, \quad M_2(x_1(t)) = 1 - M_1(x_1(t))
\]

where

\[
C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.8 & 1 \\ 2 & 1.8 \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2.6 & 0 \\ 0 & 2.5 \end{bmatrix}.
\]

The neuron activation functions are assumed to satisfy Assumption 1 with \( K = \text{diag}\{0.2, 0.2\} \).

**Solution:** For \( h_d = 0.5 \), the MADB \( \bar{h} \) that guarantees the system (18) to be asymptotically stable is calculated to be \( \bar{h} = 1.8450 \) in [1] and \( \bar{h} = 4.1560 \) in [2], which is \( \bar{h} = 5.9886 \) by using Theorem 1 in this paper. It is seen that our results improve the existing results [1, 2].

**Remark 8:** As shown in the examples, the criteria proposed in this paper have advantages over some previous ones in the sense that the computed admissible upper bound of time delay is larger. It is worth pointing out that our criteria are carried out more efficiently due to fewer matrix variables for computation.
4. Conclusion

In this paper, some less conservative LMI-based robust stability criteria are obtained without ignoring any terms in the derivative of Lyapunov-Krasovskii functional for T-S fuzzy system for neural networks with time-varying delays. Based on the Lyapunov-Krasovskii functional techniques, novel robust stability criteria have been derived in terms of linear matrix inequalities which can be easily solved using the efficient convex optimization algorithm. The LMI optimization approaches are used to obtain sufficient conditions that are very easy to be checked by using the LMI Toolbox in Matlab. Numerical examples demonstrate that the proposed method is an improvement over the existing ones.

References

7. H. Han, Adaptive fuzzy control for a class of uncertain nonlinear systems via LMI approach, International Journal of Innovative Computing, Information and


20. Y. Zhao, H. J. Gao, J. Lam, and B. Z. Du, Stability and stabilization of delayed

Appendix A. Proof of Theorem 1.

In Case I, a Lyapunov functional can be constructed as

\[ V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t), \]  

where

\[ V_1(x_t) = x^T(t)P_x(t), \]
\[ V_2(x_t) = 2\sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} g_i(s)ds + \int_{t-h(t)}^{t} x^T(s)Ug(x(s))ds, \]
\[ V_3(x_t) = \int_{t-h_1}^{t} x^T(s)Q_2x(s)ds + \int_{t-h(t)}^{t} x^T(s)Q_2x(s)ds + \int_{t-h(t)}^{t} x^T(s)Q_2x(s)ds, \]
\[ V_4(x_t) = \int_{t-h(t)}^{t} x^T(s)R_2x(s)dsd\theta + \int_{t-h(t)}^{t} x^T(s)R_2x(s)dsd\theta + \int_{t-h(t)}^{t} x^T(s)R_2x(s)dsd\theta. \]

Taking time derivative \( V(x_t) \) for \( t \in [0, \infty) \) along trajectory (9) yields

\[ \dot{V}(x_t) = \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t) + \dot{V}_4(x_t). \]

First the derivative of \( V_1(t) \) is

\[ \dot{V}_1(x_t) = \dot{x}^T(t)P_x(t) + x^T(t)P_\dot{x}(t) \]

\[ = x^T(t)(-C^T P - PC)x(t) + 2x^T(t)[Ag(x(t)) + Bg(x(t-h(t))]]. \]

Second, we get the bound of \( V_2(t) \) as

\[ \dot{V}_2(x_t) = 2g^T(x(t))S\dot{x}(t) + g^T(x(t))Ug(x(t)) - (1-h(t))g^T(x(t-h(t)))Ug(x(t-h(t))) \]
\[ \leq 2g^T(x(t))S[Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))] + g^T(x(t))Ug(x(t)) \]

\[ -(1-h_d)g^T(x(t-h(t)))Ug(x(t-h(t))). \]

Third, the bound of \( V_3(t) \) is as follows:
\[
\dot{V}_d(x_i) = x^T(t)(Q_2 + Q_3)x(t) - x^T(t - h(t))(1 - \dot{h}(t))Q_2x(t - h(t)) \\
+ x^T(t - \alpha h)(Q_2 - Q_3)x(t - \alpha h) - x^T(t - h)Q_2x(t - h) \\
\leq x^T(t)(Q_2 + Q_3)x(t) - x^T(t - h(t))(1 - h_d)Q_2x(t - h(t)) \\
+ x^T(t - \alpha h)(Q_2 - Q_3)x(t - \alpha h) - x^T(t - h)Q_2x(t - h).
\]  
(A5)

Finally, the bound of \( V_d(t) \) is as follows:

\[
\dot{V}_d(x_i) = x^T(t)[\alpha hR_1 + (1 - \alpha)hR_2 + \dot{h}(t)R_3]\dot{x}(t) - \int_{t - \alpha h}^{t - h}(s)R_3\dot{x}(s)ds \\
- \int_{t - h}^{t - h}(s)R_2\dot{x}(s)ds - (1 - \dot{h}(t))\int_{t - h}^{t - h}(s)R_2\dot{x}(s)ds \\
\leq x^T(t)[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]\dot{x}(t) - \int_{t - \alpha h}^{t - h}(s)R_3\dot{x}(s)ds \\
- \int_{t - h}^{t - h}(s)R_2\dot{x}(s)ds - (1 - h_d)\int_{t - h}^{t - h}(s)R_3\dot{x}(s)ds. 
\]  
(A6)

Now, we estimate the upper bound of the last three terms in inequality (A6) as:

\[
- \int_{t - \alpha h}^{t - h}(s)R_3\dot{x}(s)ds - \int_{t - h}^{t - h}(s)R_2\dot{x}(s)ds - (1 - \dot{h}(t))\int_{t - h}^{t - h}(s)R_2\dot{x}(s)ds \\
= - \int_{t - \alpha h}^{t - h}(s)R_3\dot{x}(s)ds - \int_{t - h}^{t - h}(s)R_2\dot{x}(s)ds - \int_{t - h}^{t - h}(s)(R_1 + (1 - h_d)R_3)\dot{x}(s)ds \\
- \int_{t - h}^{t - h}(s)(R_1 + (1 - h_d)R_3 - Z_{33})\dot{x}(s)ds - \int_{t - h}^{t - h}(s)Z_{33}\dot{x}(s)ds \\
- \int_{t - h}^{t - h}(s)X_{33}\dot{x}(s)ds - \int_{t - h}^{t - h}(s)Z_{33}\dot{x}(s)ds.
\]  
(A7)

From Lemma 1[15], if \( 0 \leq h(t) \leq \alpha h \), we obtain

\[
- \int_{t - h}^{t - h}(s)X_{33}\dot{x}(s)ds \\
\leq \int_{t - h}^{t - h}(s)X^T(t - h(t))X(t - \alpha h)\dot{x}(s)ds \\
= x^T(t - h)[\alpha hX_{11} + X_{13}]x(t - h(t)) + x^T(t - h(t))[\alpha hX_{12} - X_{13} + X^T_{23}]x(t - \alpha h) \\
+ x^T(t - \alpha h)[\alpha hX_{12} - X_{13} + X_{23}]x(t - h(t)) + x^T(t - \alpha h)[\alpha hX_{22} - X_{23} - X_{23}^T]x(t - \alpha h). 
\]  
(A8)
\[ -\int_{t-h}^{t} x^T(s)Y_{13}x(s)ds \leq x^T(t - \alpha h)[(1 - \alpha)hY_{11} + Y_{13}^T + Y_{13}]x(t - \alpha h) + x^T(t - \alpha h)[(1 - \alpha)hY_{12} - Y_{13} + Y_{23}^T]x(t - h) + x^T(t - h)[(1 - \alpha)hY_{12} - Y_{13}^T + Y_{23}]x(t - \alpha h) + x^T(t - h)[(1 - \alpha)hY_{22} - Y_{23} - Y_{23}^T]x(t - h), \quad \text{(A9)} \]

and

\[ -\int_{t-h(t)}^{t} x^T(s)Z_{33}x(s)ds \leq x^T(t)[h(t)Z_{11} + Z_{13}^T + Z_{13}]x(t) + x^T(t)[h(t)Z_{12} - Z_{13} + Z_{23}^T]x(t - h(t)) + x^T(t - h(t))h(t)Z_{12}Z_{23}^T + x^T(t - h(t))h(t)Z_{22} - Z_{23} - Z_{23}^T]x(t - h(t)) \leq x^T(t)[\alpha hZ_{11} + Z_{13} + Z_{13}]x(t) + x^T(t)[\alpha hZ_{12} - Z_{13} + Z_{23}^T]x(t - h(t)) + x^T(t - h(t))\alpha hZ_{12} - Z_{13} + Z_{23}]x(t - h(t)) + x^T(t - h(t))\alpha hZ_{22} - Z_{23} - Z_{23}^T]x(t - h(t)). \quad \text{(A10)} \]

The operator for term \( \dot{x}(t)[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]x(t) \) is as follows:

\[
\dot{x}(t)[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]x(t) = \left[-Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))\right]^T \times \\
\left[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3\right]x(t) - \left[-Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))\right] \\
= x^T(t)C^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]Cx(t) - x^T(t)C^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]Ag(x(t)) \\
-x^T(t)C^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]Bx(t-h(t)) \\
-g^T(t)A^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]C(t) \\
+g^T(t)A^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]Ag(x(t)) \\
+g^T(t)A^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]Bx(t-h(t)) \\
-g^T(t)(x(t-h(t)))A^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]C(t) \\
+g^T(t)(x(t-h(t)))A^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]Ag(x(t)) \\
+g^T(t)(x(t-h(t)))A^T[\alpha hR_1 + (1 - \alpha)hR_2 + \alpha hR_3]Bx(t-h(t))). \quad \text{(A11)}
\]

From (7) for appropriately dimensioned diagonal matrices \( \Lambda_i (i = 1, 2) \), we have

\[ -2g^T(x(t))\Lambda_1[g(x(t)) - Kx(t)] \geq 0, \quad \text{(A12)} \]

and

\[ -2g^T(x(t-h(t))\Lambda_1[g(x(t-h(t)) - Kx(t-(t-h(t)))] \geq 0, \quad \text{(A13)} \]

Substituting the above equations (A3)-(A13) into (A2), we obtain
\[
\dot{V}(t) \leq \xi^T(t) \Xi \xi(t) - \int_{t-h(t)}^{t} x^T(s)(R_1 + X_{33}) \dot{x}(s) ds - \int_{t-h(t)}^{t-\alpha h} x^T(s)(R_2 - Y_{33}) \dot{x}(s) ds,
\]
where \( \xi^T(t) = \begin{bmatrix} x^T(t) & g^T(x(t-h(t))) & x^T(t-h(t)) & x^T(t-\alpha h) & x^T(t-h) \end{bmatrix} \) and
\[
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 & 0 \\
\Xi_{12}^T & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\
\Xi_{13} & \Xi_{23}^T & \Xi_{33} & \Xi_{34} & 0 & 0 \\
\Xi_{14} & 0 & \Xi_{14}^T & \Xi_{44} & \Xi_{45} & 0 \\
0 & 0 & 0 & \Xi_{45}^T & \Xi_{55} & \Xi_{56} \\
0 & 0 & 0 & 0 & \Xi_{56}^T & \Xi_{66}
\end{bmatrix},
\]
where
\[
\Xi_{11} = -C^T P - PC + Q_1 + Q_3 + \alpha h Z_{11} + Z_{13} + Z_{13}^T + C^T [\alpha h R_1 + (1-\alpha) h R_2 + \alpha h R_3] C,
\]
\[
\Xi_{12} = PA - C^T S + K \Lambda_1 - C^T [\alpha h R_1 + (1-\alpha) h R_2 + \alpha h R_3] A,
\]
\[
\Xi_{13} = PB - C^T [\alpha h R_1 + (1-\alpha) h R_2 + \alpha h R_3] B, \Xi_{14} = \alpha h Z_{12} - Z_{13} + Z_{23}^T,
\]
\[
\Xi_{22} = SA + A^T S + U - 2 \Lambda_1 + A^T [\alpha h R_1 + (1-\alpha) h R_2 + \alpha h R_3] A,
\]
\[
\Xi_{23} = SB + A^T [\alpha h R_1 + (1-\alpha) h R_2 + \alpha h R_3] B,
\]
\[
\Xi_{33} = -(1-h)U - 2 \Lambda_2 + B^T [\alpha h R_1 + (1-\alpha) h R_2 + \alpha h R_3] B, \Xi_{34} = K \Lambda_2,
\]
\[
\Xi_{44} = -(1-h)Q_1 + \alpha h X_{11} + X_{13} + X_{13}^T + \alpha h Z_{22} - Z_{23} - Z_{23}^T,
\]
\[
\Xi_{45} = \alpha h X_{12} - X_{13} + X_{23}^T, \Xi_{55} = Q_2 - Q_1 + \alpha h X_{22} - X_{23} - X_{23}^T + (1-\alpha) h Y_{11} + Y_{13} + Y_{13}^T,
\]
\[
\Xi_{56} = (1-\alpha) h Y_{12} - Y_{13} + Y_{23}, \Xi_{66} = -Q_2 + (1-\alpha) h Y_{22} - Y_{23} - Y_{23}^T, K = \text{diag} \{k_1, k_2, \ldots, k_n\}.
\]

That is say, if \( \Xi < 0 \), \( R_1 - X_{33} \geq 0, R_2 - Y_{33} \geq 0, R_1 + (1-h)R_3 - Z_{33} \geq 0 \), then
\[
\dot{V}(t) < -\varepsilon \| x(t) \|^2
\]
for a sufficiently small \( \varepsilon > 0 \). Furthermore, (12) implies
\[
\sum_{i=1}^{n} \theta_i(z(t)) \Omega_i < 0,
\]
which is equivalent to (A14). Therefore, if LMIs (12) are feasible, the system (9) is asymptotically stable. This completes the proof. \( \square \)
Appendix B. Proof of Theorem 2.

If \( \alpha h \leq h(t) \leq h \), it gets

\[
-\int_{t-h}^{t} x^T(s) R_1 \dot{x}(s) ds - \int_{t-h}^{t-\alpha h} x^T(s) R_2 \dot{x}(s) ds - (1-h_d) \int_{t-h(t)}^{t} x^T(s) R_3 \dot{x}(s) ds
\]

\[
= -\int_{t-h}^{t} x^T(s) (R_1 + (1-h_d) R_3) \dot{x}(s) ds - \int_{t-h}^{t-\alpha h} x^T(s) (R_2 + (1-h_d) R_3) \dot{x}(s) ds
\]

\[
= -\int_{t-h}^{t} x^T(s) (R_1 + (1-h_d) R_3 - X_{33}) \dot{x}(s) ds - \int_{t-h}^{t-\alpha h} x^T(s) (R_2 + (1-h_d) R_3 - Y_{33}) \dot{x}(s) ds
\]

\[
= -\int_{t-h}^{t} x^T(s) (R_2 - Z_{33}) \dot{x}(s) ds - \int_{t-h}^{t-\alpha h} x^T(s) (R_3 - Z_{33}) \dot{x}(s) ds
\]

\[
= -\int_{t-h}^{t} x^T(s) Z_{33} \dot{x}(s) ds,
\]

(B1)

From integral inequality approach [14], notice that \( R_1 + (1-h_d) R_3 - X_{33} \geq 0 \), \( R_2 - Z_{33} \geq 0 \), and \( R_2 + (1-h_d) R_3 - Y_{33} \geq 0 \), yields

\[
-\int_{t-h}^{t} x^T(s) X_{33} \dot{x}(s) ds \leq x^T(t) \left[ \alpha h X_{11} + X_{13} \right] x(t)
\]

\[
+ x^T(t) \left[ \alpha h X_{12} - X_{13} + X_{23} \right] x(t - \alpha h)
\]

\[
+ x^T(t - \alpha h) \left[ \alpha h X_{12} + X_{13} - X_{23} \right] x(t)
\]

\[
+ x^T(t - \alpha h) \left[ \alpha h X_{22} - X_{23} + X_{23} \right] x(t - \alpha h),
\]

(B2)

and

\[
-\int_{t-h}^{t-\alpha h} x^T(s) Y_{33} \dot{x}(s) ds \leq x^T(t-\alpha h) \left[ (1-\alpha) h Y_{11} + Y_{13} + Y_{13} \right] x(t - \alpha h)
\]

\[
+ x^T(t-\alpha h) \left[ (1-\alpha) h Y_{12} - Y_{13} + Y_{23} \right] x(t - h(t))
\]

\[
+ x^T(t - h(t)) \left[ (1-\alpha) h Y_{12} - Y_{13} + Y_{23} \right] x(t - \alpha h)
\]

\[
+ x^T(t - h(t)) \left[ (1-\alpha) h Y_{22} - Y_{23} + Y_{23} \right] x(t - h(t)),
\]

(B3)

Combining (A3)-(A7) and (B1)-(B4) yields
\[ \dot{V}(x_i) \leq \xi^T(t) \Xi(t) - \int_{t - h(i)}^{t_0} \dot{x}(s)(R_1 + (1 - h_d)R_3 - X_{33})x(s)ds - \int_{t - h(i)}^{t_0} \dot{x}(s)(R_2 + (1 - h_d)R_3 - Y_{33} - Y_{33})x(s)ds - \int_{t - h(i)}^{t_0} \dot{x}(s)(R_2 - Z_{33})x(s)ds, \] 
(B5)

where \[ \Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & \Xi_{15} & 0 \\ \Xi_{12} & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\ \Xi_{13} & \Xi_{23} & \Xi_{33} & \Xi_{34} & 0 & 0 \\ 0 & 0 & \Xi_{34} & \Xi_{44} & \Xi_{45} & \Xi_{46} \\ \Xi_{15} & 0 & 0 & \Xi_{45} & \Xi_{55} & 0 \\ 0 & 0 & 0 & \Xi_{46} & 0 & \Xi_{66} \end{bmatrix} \]

and

\[ \Xi_{11} = -C^TP - PC + Q_1 + Q_3 + \alpha hX_{11} + X_{13} + X_{13}^T + C^T[\alpha hR_1 + (1 - \alpha)hR_2 + hR_3]C, \]
\[ \Xi_{12} = PA - C^TS + K\Lambda_1 - C^T[\alpha hR_1 + (1 - \alpha)hR_2 + hR_3]A, \]
\[ \Xi_{13} = PB - C^T[\alpha hR_1 + (1 - \alpha)hR_2 + hR_3]B, \]
\[ \Xi_{15} = \alpha hX_{12} - X_{13} + X_{23}^T, \Xi_{22} = SA + A^TS + U - 2\Lambda_1 - A^T[\alpha hR_1 + (1 - \alpha)hR_2 + hR_3]A, \]
\[ \Xi_{23} = SB - A^T[\alpha hR_1 + (1 - \alpha)hR_2 + hR_3]B, \]
\[ \Xi_{33} = -(1 - h_d)U - 2\Lambda_2 - B^T[\alpha hR_1 + (1 - \alpha)hR_2 + hR_3]B, \]
\[ \Xi_{34} = K\Lambda_2, \Xi_{44} = -(1 - h_d)Q_2 + (1 - \alpha)hY_{22} - Y_{23} - Y_{23}^T + (1 - \alpha)hZ_{11} + Z_{13} + Z_{13}^T, \]
\[ \Xi_{45} = (1 - \alpha)hY_{12} - Y_{13} + Y_{23}^T, \Omega_{46} = (1 - \alpha)hZ_{12} - Z_{13} + Z_{23}^T, \]
\[ \Xi_{55} = Q_2 - Q_3 + \alpha hX_{22} - X_{23} - X_{23}^T + (1 - \alpha)hY_{11} + Y_{13} + Y_{13}^T, \]
\[ \Xi_{66} = -Q_2 + (1 - \alpha)hZ_{22} - Z_{23} - Z_{23}^T, K = \text{diag}\{k_1, k_2, \ldots, k_n\}. \]

From Equation (B5) and the Schur complement of Lemma 2, it is easy to see that \[ \dot{V}(x_i) < 0 \] holds if \( R_1 + (1 - h_d)R_3 - X_{33} \geq 0, R_2 + (1 - h_d)R_3 - Y_{33} \geq 0, R_2 - Z_{33} \geq 0, \) and \( \alpha \leq h(t) \leq h. \) This completes the proof. \( \Box \)